# A FAIR SHAKE: HOW CLOSE CAN THE SUM OF n-SIDED DICE BE TO A UNIFORM DISTRIBUTION?

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ABSTRACT. Two possibly unfair n-sided dice, both labelled $1, 2, \ldots, n$, are rolled, and the sum is recorded. How should the dice's sides be weighted so that the resulting sum is closest to the uniform distribution on $2, 3, \ldots, 2n$? We answer this question by explicitly identifying the optimal pair of dice. This resolves a question raised by Gasarch and Kruskal in 1999 in a surprising way. We present additional results for the case of more than two possibly unfair n-sided dice and for the hypothetical case where the weights on each die are permitted to be negative, but must still sum to one.

1. INTRODUCTION

If you roll two six-sided dice, the probability distribution for their sums is triangle shaped:
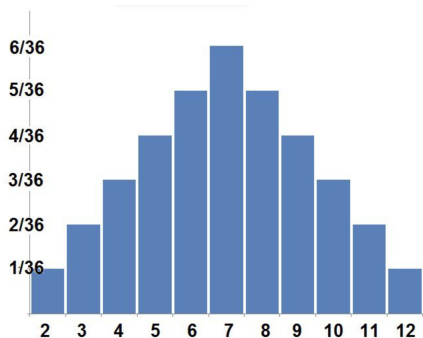
This, of course, assumes that there is an equal chance of obtaining each of the six sides. If you could change the probability of rolling each of the six sides for each of the two dice, could you obtain a uniform distribution for each sum? That is, could you replace the triangular distribution above with a rectangular distribution?

This question was posed by John Kelly in 1950 [Kel50], and subsequently it was proven via a number of methods that no matter how the two dice are weighted, a uniform distribution cannot be obtained; see, for example, [KMW+51], [Hon78], or [Hof95]. In particular, a recent book by Bollobás [Bol06] (see Problem 11) contains two solutions: one uses an elementary argument involving the arithmetic mean-geometric mean (AM-GM) inequality, and the other solution employs generating functions. Further, this was extended by Chen-Rao-Shreve [CRS97] to show that even if you have m dice, each with n sides, the dice cannot be weighted so that a uniform distribution is attained for the dice sums. More recently, Morrison [Mor18], following the main results from Gasarch and Kruskal [GK99], systematically constructed when m weighted dice — each with a potentially different numbers of sides, unlike in [CRS97] and this paper — lead to the dice sums being uniformly distributed.

Given the result from Chen-Rao-Shreve [CRS97] that a uniform distribution cannot be attained, Gasarch and Kruskal pose an interesting question at the end of their article [GK99] that we will focus on in this paper: what weightings for each of two n-sided dice will minimize D, the sum of

the squared difference between the probability for each dice sum and the uniform distribution? We note that this minimum D must be positive, since $D = 0$ corresponds to the unattainable uniform distribution.

Gasarch and Kruskal state that by programming in Matlab, they find that "[f]or all n, Matlab produced symmetric dice that were identical to each other," where the term "symmetric dice" means that the probability of rolling a 1 and rolling an n is the same, the probability of rolling a 2 and rolling an $n-1$ is the same, etc., and the term "identical to each other" means that the probability of rolling any specific side is the same for both dice. For example, for two six-sided dice, they find that Matlab minimized D when the probability of rolling a 1 or a 6 on either die is 0.243883, the probability of rolling a 2 or a 5 is 0.137480, and the probability of rolling a 3 or a 4 is 0.118637. They are, however, careful to state that "Matlab *does not* guarantee that the results are the true optimum, so the question of whether or not optimal dice must be identical and symmetric is interesting and open, even in the cases where we obtained numerical results."

In this paper, we show that Gasarch and Kruskal were right to carefully point out that their Matlab evidence was not the end of the story. In fact, their Matlab results do *not* accurately reflect the true minimum of D when $n > 2$! Specifically, we will prove the following theorem.

Theorem 1. *The two n-sided dice with side probabilities $\left(\frac{1}{2}, 0, 0, \cdots, 0, 0, \frac{1}{2}\right)$ and $\left(\frac{2}{3n-2}, \frac{3}{3n-2}, \frac{3}{3n-2}, \cdots, \frac{3}{3n-2}, \frac{3}{3n-2}, \frac{2}{3n-2}\right)$ are, up to swapping, the unique pair that minimize D, the sum-of-squares difference from the uniform distribution. This minimized value of D is $\frac{1}{2(2n-1)(3n-2)}$, which corresponds to the following probabilities for each of the dice sums: $\left(\frac{2}{2(3n-2)}, \frac{3}{2(3n-2)}, \frac{3}{2(3n-2)}, \cdots, \frac{3}{2(3n-2)}, \frac{3}{2(3n-2)}, \frac{4}{2(3n-2)}, \frac{3}{2(3n-2)}, \frac{3}{2(3n-2)}, \cdots, \frac{3}{2(3n-2)}, \frac{3}{2(3n-2)}, \frac{2}{2(3n-2)}\right)$.*

According to Theorem 1, the actual minimum of D corresponds to symmetric dice, but if $n > 2$, they are *not* identical to each other. For example, when $n = 3$, the closest to uniform distribution is obtained when one of the two dice has a half and half chance of rolling a 1 or a 3 and the other die has a $\frac{2}{7}$ chance of rolling a 1, a $\frac{3}{7}$ chance of rolling a 2, and a $\frac{2}{7}$ chance of rolling a 3.

Another potential way to obtain a uniform distribution is to allow for the possibility of negative weights to be assigned to some dice sides. That is, we still require that the weights assigned to the sides of each die add up to one, but we allow these side weights to take negative values. When does this allow us to attain a uniform distribution? The answer is given by the following theorem that we will prove in this paper:

Theorem 2. *If we have m dice where $m \geq 2$ and we allow negative weights to be assigned to sides of dice, then weightings can be found to attain a uniform distribution (i.e., $D = 0$) if and only if n, the number of dice sides on each of the m dice, is an odd number.*

In Section 2, we introduce notation needed for our proofs. We establish Theorem 1 and Theorem 2 in Section 3 and Section 4, respectively. In the spirit of Gasarch and Kruskal, in Section 5, which concludes this paper, we present two open conjectures supported by what appears to be strong numerical evidence. The easily modified Python program used to gather this numerical evidence is available in an online appendix at https://tinyurl.com/294ycnha.

2. Notation

In the sections that follow, we will use the following notation:

- m is the number of dice rolled.

- n is the number of sides on each of the m dice. The sides are labelled $1, 2, ..., n$. In general, we will use the letter i to index the sides. That is, $i = 1, 2, ..., n$.

- p_i is the probability (or weight) corresponding to the first die landing on side i. Similarly, q_i is the probability (or weight) corresponding to the second die landing on side i. In Section 5, we will also use r_i for the result of a third die. In Section 4 and the next bullet point, we will discuss m dice, so we will use the notation $p_{i_1}, q_{i_2}, ..., z_{i_m}$ to denote the probability of landing on each side of the first die, the second die,..., the mth die.

- c_j is the probability that the sum of the m dice equals j. So, for example, for two standard dice (i.e., $m = 2, n = 6$, and $p_i = q_i = \frac{1}{6}$ for $i = 1, 2, ..., 6$), we have $c_3 = \frac{2}{36}$, since there are two ways to roll a 3 (rolling 1 then 2 or rolling 2 then 1). In general, we will use the letter j to index the possible dice sums. That is, if $m = 2$, we have $j = 2, 3, ..., 2n$, and for a general m, we have $j = m, m + 1, ..., mn$. Given this, for m dice, c_j is defined by $c_j = \sum_{i_1 + i_2 + \cdots + i_m = j} p_{i_1} \times q_{i_2} \cdots \times z_{i_m}$ where $j = m, m + 1, ..., mn$.

- For two dice, D, the sum of the squared difference between the probability for each dice sum and the uniform distribution, is given by $D = \sum_{j=2}^{2n} \left(c_j - \frac{1}{2n-1} \right)^2$. For m dice, $D = \sum_{j=m}^{mn} \left(c_j - \frac{1}{m(n-1)+1} \right)^2$. Note that for two dice, the uniform distribution corresponds to $c_j = \frac{1}{2n-1}$ for $j = 2, 3, ..., 2n$, while for m dice, it corresponds to $c_j = \frac{1}{m(n-1)+1}$ for $j = m, m+1, ..., mn$.

3. Identifying the optimal dice

In this section, we will prove Theorem 1, which establishes the probabilities for each side of two n-sided dice that are needed to minimize D, and therefore obtain the most uniform distribution for the sum of the dice. We first rewrite Theorem 1 using the notation from the last section:

Theorem 1. *D is minimized if and only if either*

(1) *$p_i = 0$, except $p_1 = p_n = \frac{1}{2}$, and $q_i = \frac{3}{3n-2}$, except $q_1 = q_n = \frac{2}{3n-2}$, or*

(2) *we swap the roles of all the p_i and the q_i above, which, by symmetry, yields the same D value.*

This minimum possible value for D is $\frac{1}{2(2n-1)(3n-2)}$, which corresponds to $c_j = \frac{3}{2(3n-2)}$, except $c_2 = c_{2n} = \frac{1}{(3n-2)}$ and $c_{n+1} = \frac{2}{(3n-2)}$.

We next motivate our approach. Recall that c_j is the probability of the dice summing to j, given that p_i is the probability of rolling an i with the first n-sided die and q_i is the probability of rolling an i with the second n-sided die, so $i = 1, 2, ..., n$ and $j = 2, 3, ..., 2n$. Since the c_j are probabilities, we know that $\sum_{j=2}^{2n} c_j = 1$. Therefore, we can re-express D, the quantity that we are trying to minimize that measures the non-uniformity of the dice sum distribution, by

$$D = \sum_{j=2}^{2n} \left(c_j - \frac{1}{2n-1} \right)^2 = \sum_{j=2}^{2n} c_j^2 - 2 \sum_{j=2}^{2n} \frac{c_j}{2n-1} + \sum_{j=2}^{2n} \left(\frac{1}{2n-1} \right)^2 = \sum_{j=2}^{2n} c_j^2 - \frac{1}{2n-1}.$$

Therefore, in order to minimize D, it suffices to minimize $\sum_{j=2}^{2n} c_j^2$.

We make a key observation for understanding why the uniform distribution $c_2 = c_3 = \cdots = c_{2n}$ is unattainable using an argument that follows both the method in the first solution in Bollobás [Bol06] and the approach of Moser and Wahab in [KMW+51]. This observation centers on examining the

relationship between c_2, c_{n+1} and c_{2n}. Note that all three of these probabilities depend on p_1, q_1, p_n, and q_n. The key inequality relating c_2, c_{n+1} and c_{2n} is AM-GM, the arithmetic mean-geometric mean inequality:

Lemma 3. $c_{n+1} \geq 2\sqrt{c_2 c_{2n}}$ and equality holds if and only if $p_i q_{n+1-i} = 0$ for $2 \leq i \leq n-1$ and $p_1 q_n = p_n q_1$.

Proof. We have, by the definition of c_{n+1} and AM-GM, that

$$c_{n+1} = \sum_{i=1}^{n} p_i q_{n+1-i} \geq p_1 q_n + p_n q_1 \geq 2\sqrt{p_1 q_1 p_n q_n} = 2\sqrt{c_2 c_{2n}}.$$

In order for $c_{n+1} = 2\sqrt{c_2 c_{2n}}$, every inequality sign must be an equality. Recalling conditions for equality in AM-GM, we must have $p_i q_{n+1-i} = 0$ for $2 \leq i \leq n-1$ for the first equality and $p_1 q_n = p_n q_1$ for the second equality. $\square$

One now sees why it is impossible for all of the c_i's to have the same value: requiring that $c_2 = c_{n+1} = c_{2n} > 0$ would, from Lemma 3, imply that $1 \geq 2$! Lemma 3 also suggests a reasonable path forward. Rather than try to immediately solve for the p_i and q_i that minimize $\sum_{j=2}^{2n} c_j^2$, we should use the fact that the c_j are probabilities, in conjunction with Lemma 3, to directly find the c_j that minimize $\sum_{j=2}^{2n} c_j^2$ and then use our optimal c_j values to obtain the p_i and q_i values.

To gain a better understanding of the next step, we recall the Cauchy-Schwarz inequality in $\mathbb{R}^k$:

$$|x_1 y_1 + x_2 y_2 + \cdots + x_k y_k| \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2} \cdot \sqrt{y_1^2 + y_2^2 + \cdots + y_k^2},$$

where $x_1, \cdots, x_k, y_1, \cdots, y_k$ are any real numbers. Recall that equality in this expression holds if and only if the vectors $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ are scalar multiples of each other. One immediate consequence of Cauchy-Schwarz is that for any real $x_1, \cdots, x_k$:

$$x_1^2 + \cdots + x_k^2 \geq \frac{(x_1 + \cdots + x_k)^2}{k},$$

which is obtained by setting $y_1 = y_2 = \cdots = y_k = 1$ and squaring both sides of the Cauchy-Schwarz inequality.

To this end, we separate out the quantities c_2, c_{n+1} and c_{2n} (as we have information on how they relate), and we let Ω represent the remaining indices; that is, $\Omega = \{2, 3, \cdots, 2n\} \setminus \{2, n+1, 2n\}$. If we let $s = c_2 + c_{n+1} + c_{2n}$, then, by the Cauchy-Schwarz inequality, we have that

$$\sum_{j=2}^{2n} c_j^2 = c_2^2 + c_{2n}^2 + c_{n+1}^2 + \sum_{j \in \Omega} c_j^2$$

$$\geq c_2^2 + c_{2n}^2 + c_{n+1}^2 + \frac{1}{2n-4} \left(\sum_{j \in \Omega} c_j \right)^2 = c_2^2 + c_{2n}^2 + c_{n+1}^2 + \frac{1}{2n-4}(1-s)^2.$$

Therefore, to minimize $\sum_{j=2}^{2n} c_j^2$, it is reasonable to try to find $c_2, c_3, \cdots, c_{2n}$ so that $c_i = c_j$ whenever $i, j \in \Omega$. If we can also bound $c_2^2 + c_{2n}^2 + c_{n+1}^2$ below by an expression in $s = c_2 + c_{n+1} + c_{2n}$, then we have bounded $\sum_{j=2}^{2n} c_j^2$ below by an expression in s. To that end, we note that for any $x, y, z \geq 0$, the Cauchy-Schwarz inequality yields $x^2 + y^2 + z^2 \geq \frac{1}{3}(x + y + z)^2$. For our purpose of finding optimal dice, we use the additional stipulation $z \geq 2\sqrt{xy}$ to obtain a stronger inequality in the following lemma:

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Lemma 4. *Let $x, y, z \geq 0$, with $x + y + z = k$ and $z \geq 2\sqrt{xy}$. Then $x^2 + y^2 + z^2 \geq \frac{3}{8}(x + y + z)^2$, with equality if and only if $x = y = \frac{k}{4}$ and $z = \frac{k}{2}$.*

Proof. We have $z^2 - 4xy \geq 0$ by hypothesis. Some algebra shows that the following identity holds:

$$8(x^2 + y^2 + z^2) - 3(x + y + z)^2 = 2(z^2 - 4xy) + (z - x - y)^2 + (z - 2x)^2 + (z - 2y)^2 \geq 0.$$

The desired inequality $x^2 + y^2 + z^2 \geq \frac{3}{8}(x + y + z)^2$ immediately follows. Moreover, equality requires that $z = 2x$ and $z = 2y$, which then forces $z^2 = 4xy$ and $z = x + y$. Since $x + y + z = k$, we conclude that the equality holds if and only if $x = y = \frac{k}{4}$ and $z = \frac{k}{2}$. $\qquad\square$

In our situation, since we know that $c_{n+1} \geq 2\sqrt{c_2 c_{2n}}$, then setting $x = c_2$, $y = c_{2n}$ and $z = c_{n+1}$ gives us the inequality $c_2^2 + c_{n+1}^2 + c_{2n}^2 \geq \frac{3}{8}s^2$ with equality if and only if $c_2 = c_{2n}$ and $c_{n+1} = 2c_2$. From Lemma 3, this means that in order for the equality $c_2^2 + c_{n+1}^2 + c_{2n}^2 = \frac{3}{8}s^2$ to hold, we must have $p_i q_{n+1-i} = 0$ whenever $2 \leq i \leq n - 1$, and $p_1 q_n = p_n q_1$.

Putting this all together, we now have that

$$\sum_{j=2}^{2n} c_j^2 \;\geq\; c_2^2 + c_{2n}^2 + c_{n+1}^2 + \frac{1}{2n - 4}(1 - s)^2$$

$$\geq\; \frac{3}{8}s^2 + \frac{1}{2n - 4}(1 - s)^2,$$

with equality if and only if $c_i = c_j$ whenever $i, j \in \Omega$, $p_i q_{n+1-i} = 0$ whenever $2 \leq i \leq n - 1$, and $p_1 q_n = p_n q_1$.

The path forward is now clear: We find $s \in [0, 1]$ that minimizes $f(s) = \frac{3}{8}s^2 + \frac{1}{2n-4}(1 - s)^2$. If we could guarantee that $\sum_{j=2}^{2n} c_j^2 = \frac{3}{8}s^2 + \frac{1}{2n-4}(1 - s)^2$, then from our conditions on equality in the previous paragraph, we could then explicitly solve for each c_j. If we can find p_i's and q_i's that produce these c_j, then we have obtained our desired minimum.

Observe that the graph of $f(s) = \frac{3}{8}s^2 + \frac{1}{2n-4}(1 - s)^2$ is an upward-facing parabola whose vertex is at the value of s where $f'(s) = \frac{3}{4}s - \frac{1}{n-2}(1 - s) = 0$. That value is $s = \frac{4}{3n-2}$, which gives us

$$\sum_{j=2}^{2n} c_j^2 \geq \frac{3}{8}\frac{16}{(3n - 2)^2} + \frac{1}{2n - 4}\frac{(3n - 6)^2}{(3n - 2)^2} = \frac{3}{2(3n - 2)}.$$

For equality to hold, $c_2 = c_{2n} = \frac{s}{4} = \frac{1}{3n-2}$, $c_{n+1} = \frac{s}{2} = \frac{2}{3n-2}$ and if $j \in \Omega$, then $(2n - 4)c_j = 1 - s$. meaning that $c_j = \frac{1}{2n-4}(1 - s) = \frac{1}{2n-4}(1 - \frac{4}{3n-2}) = \frac{3}{2(3n-2)}$. We note that $\sum_{j=2}^{2n} c_j = 1$ with these values, so our c_j values for $j = 2, 3, ..., 2n$ are the unique values that minimize D.

Recapping, we have that

$$D = \sum_{j=2}^{2n}\left(c_j - \frac{1}{2n - 1}\right)^2 = \sum_{j=2}^{2n} c_j^2 - \frac{1}{2n - 1} \geq \frac{3}{2(3n - 2)} - \frac{1}{2n - 1} = \frac{1}{2(2n - 1)(3n - 2)},$$

where equality – and therefore minimizing D – can be achieved with the optimal choices of $c_2 = c_{2n} = \frac{1}{(3n-2)}$, $c_{n+1} = \frac{2}{(3n-2)}$, and $c_j = \frac{3}{2(3n-2)}$ for $j \in \Omega$.

To finish the proof of Theorem 1, we need to find the values of the p_i and q_i that will realize the values of c_j just obtained above.

Claim: To attain the optimal values of c_j above, we either have that $p_1 = p_n = 1/2$, $q_1 = q_n = 2/(3n - 2)$, and, for each $i = 2, \ldots, n - 1$, $p_i = 0$ and $q_i = 3/(3n - 2)$, or we swap the roles of

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the p_i and q_i, so $q_1 = q_n = 1/2$, $p_1 = p_n = 2/(3n-2)$, and, for each $i = 2, \ldots, n-1$, $q_i = 0$ and $p_i = 3/(3n-2)$.

Proof of claim. Recall that since $c_{n+1} = 2\sqrt{c_2 c_{2n}}$, we must have that $p_i q_{n+1-i} = 0$ for $2 \leq i \leq n-1$. In particular, it follows that $p_2 q_{n-1} = 0$. We will assume that $p_2 = 0$, instead of $q_{n-1} = 0$, and use this assumption to show that p_k must be 0 for all $k \in \{2, \cdots, n-1\}$. This observation will then determine the values of each q_i. If we assume that $q_{n-1} = 0$, instead of assuming $p_2 = 0$, a symmetric argument leads to the same results but with the values of the p_i and the q_i swapped.

Since $c_2 = p_1 q_1$ and $c_{2n} = p_n q_n$, we get:

$$p_1 q_1 + p_n q_n = c_2 + c_{2n} = c_{n+1} = p_1 q_n + p_n q_1,$$

which can be rearranged as

$$p_1 q_1 + p_n q_n - p_1 q_n - p_n q_1 = (p_1 - p_n)(q_1 - q_n) = 0,$$

meaning either $p_1 = p_n$ or $q_1 = q_n$. However, $p_1 q_1 = p_n q_n$ (since $c_2 = c_{2n}$), so each of these equalities forces the other to hold, meaning

$$p_1 = p_n \text{ and } q_1 = q_n.$$

Next, using that $p_2 = 0$, we look at $c_3 = p_1 q_2 + p_2 q_1 = p_1 q_2$. Since $c_3 = \frac{3}{2} c_2 = \frac{3}{2} p_1 q_1$, we obtain: $\frac{3}{2} p_1 q_1 = p_1 q_2$. which implies that $q_2 = \frac{3}{2} q_1$.

Recall that we must have that $p_i q_{n+1-i} = 0$ for $2 \leq i \leq n-1$. In particular, setting $i = n-1$, we have that $p_{n-1} q_2 = 0$ and so, since $q_2 > 0$, we have that $p_{n-1} = 0$.

Recapping, we have that $p_1 = p_n$ and $q_1 = q_n$, which we will use at the end of our proof, and we have that $p_2 = p_{n-1} = 0$ and $q_2 = \frac{3}{2} q_1 > 0$, which will form the initial step for the following induction argument:

> **Induction step:** For $i \in \{1, 2, ..., n-3\}$, if we have that $p_{n-1} = p_{n-2} = ... = p_{n-i} = 0$, then $q_{n-i} = \frac{3}{2} q_1$ and $p_{n-i-1} = 0$.
>
> *Proof of the induction step:* Combining $c_{2n-i} = \frac{3}{2} c_2 = \frac{3}{2} p_1 q_1$ and the definition of c_{2n-i}, we have:
>
> $$\frac{3}{2} p_1 q_1 = c_{2n-i} = p_n q_{n-i} + p_{n-1} q_{n-i+1} + ... + p_{n-i} q_n.$$
>
> Given that $p_{n-1} = p_{n-2} = ... = p_{n-i} = 0$, the above equality implies that $\frac{3}{2} p_1 q_1 = p_n q_{n-i} = p_1 q_{n-i}$, which means $q_{n-i} = \frac{3}{2} q_1$.
>
> On the other hand, combining $c_{n-i+1} = \frac{3}{2} c_2 = \frac{3}{2} p_1 q_1$ and the definition of c_{n-i+1}, we have:
>
> $$\frac{3}{2} p_1 q_1 = c_{n-i+1} = p_1 q_{n-i} + p_2 q_{n-i-1} + ... + p_{n-i-1} q_2 + p_{n-i} q_1.$$
>
> Since $q_{n-i} = \frac{3}{2} q_1$, we see that $p_1 q_{n-i}$ term above is already $\frac{3}{2} p_1 q_1$, which means the other terms, being non-negative, must each be zero. In particular, $p_{n-i-1} q_2 = 0$, which forces $p_{n-i-1} = 0$ since $q_2 > 0$. $\square$

With the induction step established, we can run it using $i = 1$, then $i = 2$, $i = 3$, and so on, until we stop at $i = n-3$. This establishes that $p_i = 0$ and $q_i = \frac{3}{2} q_1$ for $i = 2, 3, ..., n-1$.

Since we now know that $p_1 = p_n$ and all other $p_i = 0$, we have from $\sum_{i=0}^{n} p_i = 1$ that $p_1 = p_n = \frac{1}{2}$. Similarly, since we now know that $q_1 = q_n$ and all other $q_i = \frac{3}{2} q_1$, we have from $\sum_{i=0}^{n} q_i = 1$ that $q_1 = q_n = \frac{2}{3n-2}$ and all other $q_i = \frac{3}{3n-2}$, which establishes our claim. $\square$

And with our claim established, this completes the proof of Theorem 1. $\qquad\square$

Remark 5. Our argument shows that the minimum value of D, which measures the distance from the uniform distribution on the set $\{2,\ldots,2n\}$, approaches 0 as $n \to \infty$. In other words, even though the uniform distribution is impossible to achieve using the sum of two dice, the best approximation using the optimal dice converges to the desired uniform distribution at a rate of $O\left(\frac{1}{n^2}\right)$ (since $D = \frac{1}{2(2n-1)(3n-2)}$) as n, the number of sides, tends to infinity.

Remark 6. Also regarding convergence, note that if X and Y are independent random variables where X attains the values 0 and 1 with equal probability (of $1/2$) and Y is uniformly distributed on the interval $[0,1]$, then the distribution of $X + Y$ is uniform on $[0,2]$. This means, from our discrete dice perspective, if we let X_n and Y_n be random variables corresponding to rolling each of the two optimal dice given in Theorem 1, except that we normalize the die faces to be $1/n, 2/n, \ldots, 1$, then as $n \to \infty$, X_n and Y_n converge in distribution to X and Y. This fits with the flavor of our results and helps further indicate why the optimal dice were unlikely to be symmetric (that is, to have $p_i = q_i$ for all $i = 1, 2, \ldots, n$).

4. An odd result: generating the proof of Theorem 2

In this section, we will prove Theorem 2, which shows that for $m \geq 2$ dice, allowing negative values for the $p_i, q_i, \ldots, z_i$ makes a uniform distribution for dice sums attainable if and only if n is odd. More precisely, using the notation introduced in Section 2, we can rewrite Theorem 2 as:

Theorem 2. *For m dice, allow $p_i, q_i, \ldots, z_i$, where $i = 1, 2, \ldots, n$, to be any real (possibly negative) numbers, still subject to the restriction that $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \ldots = \sum_{i=1}^{n} z_i = 1$. Each c_j is still defined by $c_j = \sum_{i_1+i_2+\cdots+i_m=j} p_{i_1} \times q_{i_2} \times \cdots \times z_{i_m}$ for $j = m, m+1, \ldots, mn$. If and only if n is odd, $p_i, q_i, \ldots, z_i$ can be chosen so that $c_j = \frac{1}{m(n-1)+1}$ for all j, and therefore $D = \sum_{j=m}^{mn} \left(c_j - \frac{1}{m(n-1)+1}\right)^2 = 0$. In other words, if n is odd, then a uniform distribution for the c_j is possible. If n is even, then a uniform distribution for the c_j is not possible.*

Proof. If a set of m dice with n sides has uniform totals, then the generating function of the totals is $\frac{1}{m(n-1)+1}(x^m + \ldots + x^{mn}) = \frac{x^m}{m(n-1)+1}T(x)$, where $T(x) = 1 + x + \cdots + x^{m(n-1)}$. Note that $T(x)$ has roots at the non-trivial $(m(n-1)+1)$st roots of unity; in particular, it has a single root -1 or none according to whether $m(n-1)+1$ is even or odd. Now, the generating function of *each* die is of the form $p_1 x + p_2 x^2 + \cdots + p_n x^n = x(p_1 + p_2 x + \cdots + p_n x^{n-1})$ for some p_i; the second factor must have a real root if n is even. Thus, if n is even, and $m > 1$, the product of the dice generating functions cannot equal $T(x)$. If n is odd, then any partition of $m(n-1)/2$ real quadratic factors of $T(x)$ into m sets of size $\frac{n-1}{2}$ yields m generating functions of degree $n-1$ with product $T(x)$. Multiplying each of these polynomials by x, and normalizing the polynomials' coefficients appropriately, we obtain m generating functions whose coefficients give us the desired probabilities. $\qquad\square$

5. Numerical results create two new open questions for multiple dice

The numerical experiments conducted by Gasarch and Kruskal [GK99] for $m = 2$ dice inspired us to resolve the open questions they posed. Similarly, our numerical experiments have led to two new open questions for multiple dice that we hope others will explore. Our numerical experiments for m dice were conducted in Python. The Jupyter notebook containing the Python code for the case

of three n-sided dice is available in an online appendix, which is located at https://tinyurl.com/294ycnha.[1]

First open question: For two dice, Gasarch and Kruskal suggested that the n-sided dice that minimized D had to be *symmetric*, meaning for the first die that $p_1 = p_n$, $p_2 = p_{n-1}$, $p_3 = p_{n-2}$, etc., and, similarly, for the second die that $q_1 = q_n$, $q_2 = q_{n-1}$, $q_3 = q_{n-2}$, etc. This paper proves that conjecture to be true when $m = 2$ dice, but it is an open question as to whether or not this holds for $m > 2$ dice. Our numerical experiments always produce symmetric dice results for minimizing D, but that is not a proof, of course!

Second open question: In fact, our experimental results go farther than this, indicating a specific symmetric pattern, which is that the quantity D is minimized when the probabilities on one die (say, the first die, so we are specifying the p_i) are given by

$$p_i = \begin{cases} \frac{m}{(n-2)(2m-1)+2m} & \text{if } i = 1 \text{ or } n \\ \frac{2m-1}{(n-2)(2m-1)+2m} & \text{if } i = 2, 3, ..., n-1, \end{cases}$$

while, for the other $m - 1$ dice, we have

$$q_i = r_i = ... = \begin{cases} \frac{1}{2} & \text{if } i = 1 \text{ or } n \\ 0 & \text{if } i = 2, 3, ..., n-1. \end{cases}$$

For example, in our online appendix, we see that the results for $m = 3$ and $n = 5$ (three five-sided dice) fit this pattern. That is,

$$(p_1, p_2, p_3, p_4, p_5) = \left(\frac{3}{21}, \frac{5}{21}, \frac{5}{21}, \frac{5}{21}, \frac{3}{21} \right)$$

$$(q_1, q_2, q_3, q_4, q_5) = \left(\frac{1}{2}, 0, 0, 0, \frac{1}{2} \right)$$

$$(r_1, r_2, r_3, r_4, r_5) = \left(\frac{1}{2}, 0, 0, 0, \frac{1}{2} \right).$$

The above symmetric pattern agrees with the results of every other example we ran as well, using a variety of values for n and m, but the proof (or refutation) of this pattern when $m > 2$ remains an open question.

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[1]The code works with probabilities p_i, q_i, and r_i for the three dice (labelled `p_i`, `q_i`, and `r_i` in the code), where $i = 1, 2, ..., n$. The code is easily extended to more than three dice — or contracted for the case of two dice. To remove the restriction that each p_i, q_i, and r_i must be between 0 and 1, we can simply remove the phrase "bounds=bounds" from the minimize command in code block [6]. However, our open questions in this section consider actual probabilities, so we keep the restriction of being between 0 and 1.

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