

COLORING GRAPHS AS COMPLETE GRAPH INVARIANTS

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ABSTRACT. We investigate the extent to which the k -coloring graph $\mathcal{C}_k(G)$ uniquely determines the base graph G and the number of colors k . The vertices of $\mathcal{C}_k(G)$ are the proper k -colorings of G , and edges connect colorings that differ on exactly one vertex. There are nonisomorphic graphs G_1 and G_2 with isomorphic coloring graphs, so $\mathcal{C}_k(G)$ is not a complete invariant in general. However, for color palettes with *surplus colors* (when the number of colors k is greater than the chromatic number), we prove that the coloring graph is a complete invariant. Specifically, provided that $k_1 > \chi(G_1)$, we show that $\mathcal{C}_{k_1}(G_1) \cong \mathcal{C}_{k_2}(G_2)$ implies $G_1 \cong G_2$ and $k_1 = k_2$. Thus, there is a natural bijection between pairs (G, k) with $k > \chi(G)$ and their coloring graphs $\mathcal{C}_k(G)$. Furthermore, no coloring graph of the form $\mathcal{C}_{\chi(G)}(G)$ is isomorphic to a coloring graph with surplus colors. Our constructive proof provides a method to decide whether a coloring graph is generated with surplus colors, although the resulting algorithms are inefficient.

1. INTRODUCTION

A graph $G = (V, E)$ is a fundamental structure that captures pairwise relationships between objects. Two graphs are *isomorphic* if an adjacency-preserving bijection exists between their vertex sets. Determining isomorphism is computationally challenging, motivating the study of *graph invariants*, properties preserved under isomorphism. Graph colorings provide a rich source of various graph invariants. A *proper k -coloring* is a function $c: V \rightarrow \{1, \dots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. The minimum number k required for such a coloring is the *chromatic number*, denoted $\chi(G)$. Beyond the threshold for which a proper coloring exists, we can also count the number of proper k -colorings as a function of k and obtain the *chromatic polynomial* $\pi_G(k)$. Rather than counting the proper k -colorings in isolation, we can investigate how two k -colorings are related using the *k -coloring graph*, denoted $\mathcal{C}_k(G)$. The vertices of $\mathcal{C}_k(G)$ represent proper k -colorings of G , and two vertices are adjacent if the corresponding colorings differ in the color assigned to exactly one vertex $v \in V$. This construction places $\mathcal{C}_k(G)$ within the broader context of a *reconfiguration graph*, where edges represent a minimal transition between valid states.

A natural question arises: to what extent does $\mathcal{C}_k(G)$ encode the structure of the base graph G ? Specifically, under what conditions can $\mathcal{C}_k(G)$ serve as a complete graph invariant, uniquely determining G (up to isomorphism) and potentially the palette size k ? If $k < \chi(G)$, then $\mathcal{C}_k(G)$ is the null graph N_0 (the graph with no vertices), providing no information. If $k = \chi(G)$, the structure can vary; for instance, consider a *uniquely k -colorable* graph which has a unique partition into $k = \chi(G)$ independent sets. Every uniquely k -colorable graph G yields $\mathcal{C}_k(G) \cong N_{k!}$, representing the $k!$ permutations of the unique coloring pattern as isolated vertices (where N_r denotes the graph with r vertices and 0 edges). This paper investigates the critical role of the palette size k , particularly when it exceeds the chromatic number $\chi(G)$ (that is, when there are *surplus colors*), in determining whether $\mathcal{C}_k(G)$ uniquely identifies G . To measure complete invariance of $\mathcal{C}_k(G)$, in joint work [AKLR24] with Russell, we posed the following problem:

Does there exist a function $f: \text{Graphs} \rightarrow \mathbb{N}$ such that G can be uniquely reconstructed from $\mathcal{C}_{f(G)}(G)$?

Hogan, Scott, Tamitegama, and Tan [HSTT24] answered this question affirmatively by showing that the function $f(G) = 5|V(G)|^2 + 1$ satisfies the requirement. We refine their result by proving that $f(G) = \chi(G) + 1$ suffices. Our main theorem asserts an even stronger conclusion: coloring graphs with surplus colors are distinguishable from all other coloring graphs, including those of the form $\mathcal{C}_{\chi(G)}(G)$.

Theorem 1.1. *If an abstract graph \mathcal{C} satisfies $\mathcal{C} \cong \mathcal{C}_k(G)$ for some graph G with $k > \chi(G)$, then for any graph G' and positive integer k' with $\mathcal{C} \cong \mathcal{C}_{k'}(G')$, we have $G \cong G'$ and $k = k'$.*

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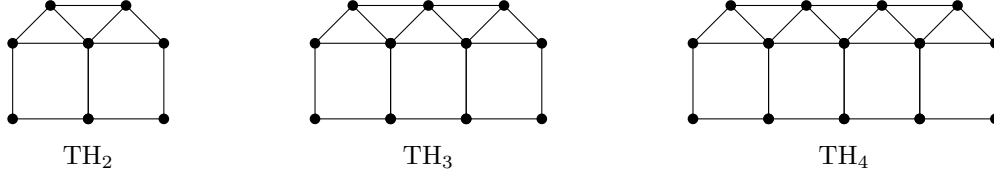
We emphasize that Theorem 1.1 is logically equivalent to the combination of two separate results:

- (a) If $\mathcal{C}_{k_1}(G_1) \cong \mathcal{C}_{k_2}(G_2)$ with $k_1 > \chi(G_1)$ and $k_2 > \chi(G_2)$, then $G_1 \cong G_2$ and $k_1 = k_2$.
- (b) No two graphs G and G' can satisfy $\mathcal{C}_k(G) \cong \mathcal{C}_{k'}(G')$ with $k > \chi(G)$ and $k' = \chi(G')$.

Part (b) turns out to be much more difficult than (a). We prove (a) in Theorem 4.4 using a reconstruction procedure outlined in Algorithm 1. Proving (b) (in Theorem 4.28) requires two additional algorithms and several other novel ideas. We also exhibit that $k = \chi(G)$ is not sufficient for $\mathcal{C}_k(G)$ to be a complete invariant by proving the following result.

Theorem 1.2. *For every $\chi_1, \chi_2 \geq 3$, there exist graphs G_1, G_2 with $\chi(G_1) = \chi_1$ and $\chi(G_2) = \chi_2$ such that $\mathcal{C}_{\chi_1}(G_1) \cong \mathcal{C}_{\chi_2}(G_2)$.*

The result is obtained by constructing a family of n -townhouse graphs for $n \geq 1$. The graph TH_n has chromatic number 3, and $\mathcal{C}_3(\text{TH}_n)$ is a disjoint union of six path graphs, each on $n + 2$ vertices.



Further modifications to these townhouse graphs for arbitrary $\chi_1, \chi_2 \geq 3$ yield distinct graphs G_1, G_2 with chromatic numbers χ_1 and χ_2 , respectively, such that $\mathcal{C}_{\chi_1}(G_1) \cong \mathcal{C}_{\chi_2}(G_2)$.

The strategy behind proving Theorem 1.1 involves identifying and analyzing specific structures within coloring graphs. We call a coloring graph $\mathcal{C}_{\chi(G)}(G)$ a χ -coloring graph. If the number of colors k exceeds $\chi(G)$, then $\mathcal{C}_k(G)$ is called a *surplus coloring graph*. The main goal of this paper is to provide a structural characterization of these latter graphs. We identify a structure called an *abstract link vertex* whose presence can be algorithmically checked in an arbitrary coloring graph (without knowing G or even k). Crucially, we show that an arbitrary coloring graph \mathcal{C} contains an abstract link vertex if and only if \mathcal{C} is generated by a unique pair (G, k) with $k > \chi(G)$. Link vertices refine the notion of rainbow colorings from [HSTT24], allowing us to establish coloring graphs as a complete invariant for k as small as $\chi(G) + 1$ rather than requiring k to be a quadratic function of $|V(G)|$. Our first algorithm, inspired by the square-counting technique in [HSTT24], reconstructs a base graph's adjacency matrix from any coloring graph with an abstract link vertex. The second algorithm extracts vertex partition information at a link vertex. The third algorithm then organizes an entire equivalence class of link vertices into a new structure called a *labeled link graph*. Collectively, these algorithms impose specific constraints on any base graph that yields the coloring graph. This leads us to conclude that a coloring graph containing a link vertex is generated by at most one pair (G, k) with $k > \chi(G)$, and by no pair (G', k') where $k' \leq \chi(G')$.

Remark 1.3. We note that Theorem 4.4 (a weaker version of Theorem 1.1) was independently discovered by Berthe, Brosse, Hearn, van den Heuvel, Hoppenot, and Pierron, as presented in their arXiv preprint [BBH⁺25], posted on April 28, 2025. We posted the first version of our paper (arXiv:2504.20978) on April 29, 2025. We were unaware of their results as this manuscript was being finalized. The approach for reconstructing the base graph is similar, but the two papers also diverge in other aspects. Specifically, our Theorem 1.1 is stronger in its conclusion. Namely, [BBH⁺25, Theorem 1.2] proves part (a), while our methods establish both part (a) and part (b), with the proof of part (b) being substantially more involved and forming the bulk of this paper. Moreover, Theorem 1.2 and its proof are novel to this work. On the other hand, Berthe et al. establish the insufficiency of $k = \chi(G)$ by constructing an arbitrarily large collection \mathcal{F} of nonisomorphic graphs all with chromatic number χ and no frozen vertices such that $\mathcal{C}_\chi(G_1) \cong \mathcal{C}_\chi(G_2)$ for all $G_1, G_2 \in \mathcal{F}$ provided that $\chi \geq 6$ [BBH⁺25, Theorem 1.4]. Their paper also contains further results concerning reconfiguration graphs of Kempe-colorings and independent sets.

Structure of the paper. Section 2 reviews fundamental definitions and introduces our notation. We prove Theorem 1.2 in Section 3. The proof of our main result, Theorem 1.1, is developed in Section 4, which is organized into four subsections with a series of supporting lemmas. In Section 4.1, we introduce Algorithm 1 and prove Theorem 4.4 (part (a) of Theorem 1.1). Section 4.2 and Section 4.3 introduce Algorithm 2 and Algorithm 3, respectively. The final Section 4.4 proves Theorem 4.28 (part (b) of Theorem 1.1).

2. DEFINITIONS AND NOTATION

Given a graph G and $k \geq \chi(G)$, the k -coloring graph of G , denoted $\mathcal{C}_k(G)$, is defined as follows. The vertices of $\mathcal{C}_k(G)$ represent proper k -colorings of G . Two vertices in $\mathcal{C}_k(G)$ (representing two distinct k -colorings) are adjacent if the two colorings differ on exactly one vertex $v \in V(G)$. Formally, we have a bijection $\Phi: V(\mathcal{C}_k(G)) \rightarrow \{\text{proper } k\text{-colorings of } G\}$ defined by $\Phi(\alpha) = \alpha_G$ such that $\alpha\beta \in E(\mathcal{C}_k(G))$ if and only if $\alpha_G(v) \neq \beta_G(v)$ for exactly one $v \in V(G)$. That is, we use α to denote a vertex in $\mathcal{C}_k(G)$ and $\alpha_G: V(G) \rightarrow [k]$ for the corresponding coloring of G . For simplicity, we typically omit the symbol Φ and use the shorthand $\alpha \mapsto \alpha_G$ for this mapping.

While $\mathcal{C}_k(G)$ is fundamentally an abstract graph, we can use the underlying bijection between $V(\mathcal{C}_k(G))$ and the k -colorings of G to produce an *edge-labeled* coloring graph. Each edge is labeled to show the unique vertex $v \in V(G)$ where the two corresponding colorings differ and the specific colors assigned to v by each coloring. For example, an edge $\alpha\beta \in E(\mathcal{C}_k(G))$ might have the label cvc' , which signifies that $\alpha_G(v) = c, \beta_G(v) = c'$, and $\alpha_G(u) = \beta_G(u)$ for all $u \in V(G) \setminus \{v\}$. We emphasize that we have access to neither edge labelings nor the $\alpha \mapsto \alpha_G$ mapping when given an abstract coloring graph $\mathcal{C} \cong \mathcal{C}_k(G)$.

Figure 1 depicts the relationship between 3-colorings of P_3 by labeling the vertices and edges of $\mathcal{C} \cong \mathcal{C}_3(P_3)$. The red vertices in \mathcal{C} , which use the minimum number of colors and are termed *abstract link vertices*, play a central role in our paper. In general, when a graph G has d connected components, the coloring graph $\mathcal{C}_k(G)$ with $k > \chi(G)$ admits a natural action of S_k^d , the d -fold product of the symmetric group S_k on k symbols. We will show that the orbit-equivalent link vertices in the coloring graph are connected via an identifiable sequence of hypercubes. For $\mathcal{C}_3(P_3)$, the six link vertices form a single equivalence class. A *labeled link graph* organizes a link vertex equivalence class into a new graph whose edges are hypercubes in $\mathcal{C}_k(G)$. These graphs, formalized in Section 4, help establish P_3 as the unique base graph yielding \mathcal{C} as a coloring graph.

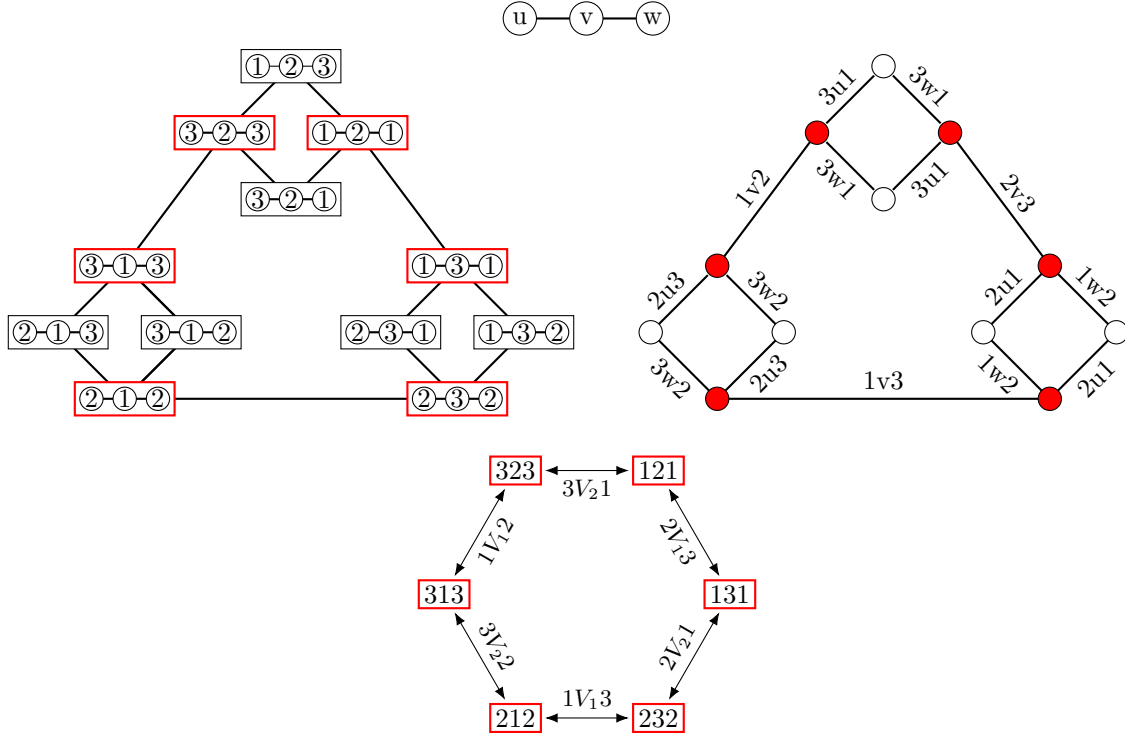


FIGURE 1. Three organizations of the 3-colorings of P_3 , with link colorings highlighted red. Top left is coloring graph $\mathcal{C}_3(P_3)$ with vertex labels indicating the underlying bijection $\Phi: V(\mathcal{C}_3(P_3)) \rightarrow \{\text{proper 3-colorings of } P_3\}$; top right depicts the corresponding edge labelings; bottom center presents the labeled link graph organizing the 2-colorings consistent with bipartition $V_1 = \{v\}, V_2 = \{u, w\}$ of $V(P_3)$.

We use standard notation for common graphs (see [Wes01]): K_n , C_n , and P_n are the complete, cycle, path graphs on n vertices; $K_{r,s}$ is the complete bipartite graph with two parts of cardinality r and s ; N_r is the graph with r vertices and 0 edges. The *house graph* is obtained by adding a chord to a 5-cycle.

Next, we describe notation related to integer and set partitions. Let \mathbb{N} denote the set of positive integers. For $k \in \mathbb{N}$, let $[k]$ be the set $\{1, 2, \dots, k\}$. A *partition* $\lambda = (\lambda_1, \dots, \lambda_t)$ of $n \in \mathbb{N}$, denoted $\lambda \vdash n$, means that $\lambda_i \in \mathbb{N}$ for all i and $\lambda_1 + \dots + \lambda_t = n$. For example, $(3, 1, 7, 1, 6) \vdash 18$.

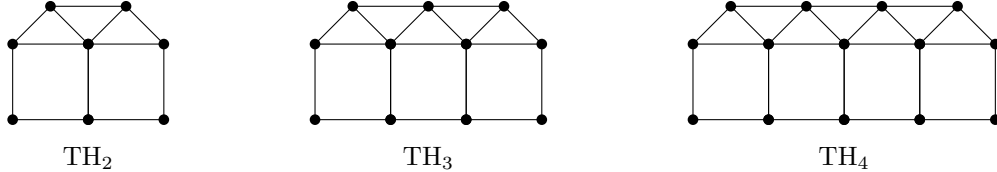
For two disjoint subsets A and B of a set X , their union $A \cup B$ is denoted by $A \uplus B$ to emphasize disjointness. A *set partition* of X is a tuple (X_1, \dots, X_t) of pairwise disjoint subsets whose union is X , so $X = X_1 \uplus X_2 \uplus \dots \uplus X_t$. By convention, each part X_i is nonempty. It is sometimes convenient to initially define parts that may be empty; such empty parts are then disregarded in the final partition. For instance, a proper k -coloring $\alpha: V(G) \rightarrow [k]$ of a graph G induces a partition $P = (P_1, P_2, \dots, P_k)$ of $V(G)$ where each part $P_c = \alpha^{-1}(c)$ consists of vertices colored $c \in [k]$. If the coloring α uses fewer than k colors, some parts P_c in this partition are empty. The number of nonempty parts in a set partition P is its *length*, $\ell(P)$.

3. COLORING GRAPHS AT THE CHROMATIC LEVEL MAY HAVE MULTIPLE BASE GRAPHS

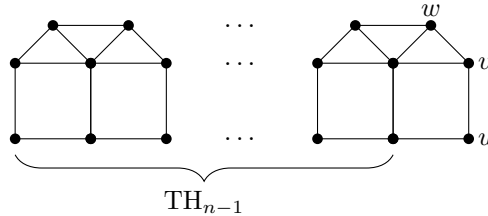
Our main result, established in Section 4, is twofold: first, surplus coloring graphs uniquely determine their base graphs; second, surplus coloring graphs are structurally distinct from χ -coloring graphs. In contrast, two nonisomorphic graphs can produce the same χ -coloring graphs. For instance, all uniquely k -colorable graphs share the same k -coloring graph, namely the graph $N_{k!}$ consisting of $k!$ isolated vertices. This observation leads to a natural question: can $\mathcal{C}_{\chi(G_1)}(G_1)$ be isomorphic to $\mathcal{C}_{\chi(G_2)}(G_2)$ when $\chi(G_1) \neq \chi(G_2)$? In this section, we affirmatively answer this question by proving Theorem 1.2. We begin by constructing a 3-chromatic graph with a prescribed number of vertices in its 3-coloring graph.

Lemma 3.1. Let $r \geq 1$. There exists a graph G with $\chi(G) = 3$ such that $\mathcal{C}_3(G)$ has exactly $6r$ vertices.

Proof. For the cases $r = 1$ and $r = 2$, the graphs K_3 and $K_{1,3} + e$ (the *paw graph*) provide the required examples, as can be verified by evaluating their chromatic polynomials at $k = 3$. Assume $r \geq 3$. Consider the *townhouse graph* TH_n for $n \geq 1$, which consists of a sequence of n 5-node house graphs in which consecutive houses share a wall and have an additional edge connecting the tops of their gables. By definition, TH_1 is the house graph. Below, we illustrate townhouse graphs for $n = 2, 3, 4$.



We claim that $\mathcal{C}_3(\text{TH}_n)$ contains precisely $6(n+2)$ vertices. By letting $n = r - 2$, this completes the proof of the lemma. When constructing TH_n from TH_{n-1} , three new vertices u, v, w are added, with v and w serving as the *roof vertices* as pictured below.



We compute the number of 3-colorings of TH_n . First, there are 3 choices for the color of vertex v and 2 choices for the color of vertex w . Once these colors are assigned, the colors of all the roof vertices (three per roof) are determined automatically. Hence, the number of 3-colorings of TH_n equals $6 \cdot t_n$, where t_n represents the number of 3-colorings of the path graph P_{n+1} with specific constraints described below.

Without loss of generality, suppose v is colored 1 and w is colored 3. The *floor vertices* of the houses form a path graph P_{n+1} , consisting of $u_{n+1} - u_n - \dots - u_2 - u_1$ where $u = u_1$. The value t_n is the number of

3-colorings of P_{n+1} such that the color of u_i differs from $i \pmod{3}$ for every $1 \leq i \leq n+1$. We use the set $\{1, 2, 3\}$ as representatives modulo 3 to correspond to the color labels.

We claim that $t_n = n + 2$. To prove this formula by induction on n , it suffices to show that $t_1 = 3$ and that $t_n = t_{n-1} + 1$ for all $n \geq 2$.

Base case. For $n = 1$, we count the colorings c of $P_2 = u_2 - u_1$ such that $c(u_1) \neq 1$ and $c(u_2) \neq 2$. If $c(u_2) = 3$, then $c(u_1) = 2$ is the only valid choice. If $c(u_2) = 1$, then $c(u_1)$ can be either 2 or 3. Thus, there are 3 valid colorings, giving $t_1 = 3$.

Recursive formula. By definition, t_{n-1} is the number of valid colorings of $P_n = u_{n+1} - u_n - \dots - u_2$. We extend these colorings to P_{n+1} while ensuring that $c(u_1) \neq 1$. The t_{n-1} valid colorings of P_n split into two groups:

- **Type A:** $c(u_2) = 3$. In this case, $c(u_1) = 2$ is the only possible extension, providing a unique valid coloring of P_{n+1} .
- **Type B:** $c(u_2) = 1$. The constraints force a specific coloring pattern on u_3, \dots, u_{n+1} , determined by $c(u_i) = (i - 1) \pmod{3}$. However, for u_1 , we have two possible extensions: $c(u_1) = 2$ or $c(u_1) = 3$.

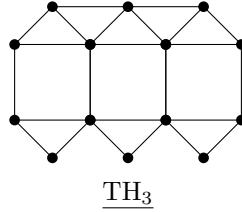
Thus, there is one additional valid coloring of Type B, giving $t_n = t_{n-1} + 1$. By induction, $t_n = n + 2$ for all $n \geq 1$. Consequently, TH_n has a total of $6(n + 2)$ distinct 3-colorings for each $n \geq 1$. \square

The proof of Lemma 3.1 can be adapted to fully describe the 3-coloring graph of townhouse graphs. For $n \geq 1$, the graph $\mathcal{C}_3(\text{TH}_n)$ is isomorphic to $6P_{n+2}$, a disjoint union of 6 path graphs, each on $n + 2$ vertices.

Corollary 3.2. For each $r \geq 1$, there exists a connected graph G with $\chi(G) = 3$ such that $\mathcal{C}_3(G) = N_{6r}$.

Proof. The cases $r = 1$ and $r = 2$ are addressed by considering the graphs K_3 and F_2 (the *friendship graph*), which consists of two copies of K_3 joined at a common vertex.

Assume $r \geq 3$ and set $n = r - 2 \geq 1$. We modify the townhouse graph TH_n from the proof of Lemma 3.1 by attaching new *basement vertices* to each house to ensure that every vertex belongs to a triangle. This construction yields a new graph $\underline{\text{TH}}_n$, illustrated below for $n = 3$.



The addition of the basement vertices does not affect the number of 3-colorings as their colors are determined automatically. Consequently, $\mathcal{C}_3(\underline{\text{TH}}_n)$ still contains $6(n + 2) = 6r$ vertices. Furthermore, because every vertex participates in a triangle, $\mathcal{C}_3(\underline{\text{TH}}_n)$ has no edges, completing the proof. \square

Corollary 3.3. For each $r \geq 1$ and $k \geq 3$, there exists a connected graph with $\chi(G) = k$ such that $\mathcal{C}_k(G) = N_{k! \cdot r}$.

Proof. We proceed by induction on k . The base case $k = 3$ follows from Corollary 3.2. For the inductive step, assume there exists a graph G with $\chi(G) = k$ such that $\mathcal{C}_k(G)$ consists of $k! \cdot r$ isolated vertices. Consider a new graph G' formed by adding a new vertex v and connecting v to all vertices of G . Then $\chi(G') = k + 1$ and $\mathcal{C}_{k+1}(G')$ consists of $(k + 1) \cdot (k! \cdot r) = (k + 1)! \cdot r$ isolated vertices. \square

We are now ready to prove one of the theorems from the introduction.

Proof of Theorem 1.2. The case $\chi_1 = \chi_2 = k$, mentioned at the beginning of this section, is straightforward. Taking $G_1 = K_k$ and $G_2 = K_{k+1} - e$, we obtain $\mathcal{C}_k(G_1) \cong \mathcal{C}_k(G_2) \cong N_{k!}$. Thus, we assume $\chi_2 > \chi_1 \geq 3$. Let $r = \frac{\chi_2!}{\chi_1!} \in \mathbb{N}$. By Corollary 3.3, there exists a graph G_1 such that $\chi(G_1) = \chi_1$ and $\mathcal{C}_{\chi_1}(G_1)$ consists of $\chi_1! \cdot r = \chi_2!$ isolated vertices. Let $G_2 = K_{\chi_2}$ which satisfies $\chi(G_2) = \chi_2$, and $\mathcal{C}_{\chi_2}(G_2)$ also consists of $\chi_2!$ isolated vertices. As $\chi(G_1) \neq \chi(G_2)$, the graphs G_1 and G_2 are nonisomorphic, but $\mathcal{C}_{\chi_1}(G_1) \cong \mathcal{C}_{\chi_2}(G_2)$. \square

We discuss another consequence of Corollary 3.3. While $\mathcal{C}_{\chi(G)}(G)$ does not generally allow for the unique reconstruction of the base graph G , it is worth considering what information about the base graph can still be derived from its χ -coloring graph. The next observation, which strengthens Theorem 1.2, shows that χ -coloring graph cannot even detect connectivity of its base graph.

Theorem 3.4. *For $\chi_2 \geq \chi_1 \geq 3$, there exists a disconnected χ_1 -chromatic graph G_1 and a connected χ_2 -chromatic graph G_2 such that $\mathcal{C}_{\chi_1}(G_1) \cong \mathcal{C}_{\chi_2}(G_2)$.*

Proof. By Corollary 3.3 applied with $k = \chi_1$ and $r = \chi_2!$, there exists a graph H with $\chi(H) = \chi_1$ such that $\mathcal{C}_{\chi_1}(H) = N_{\chi_1! \cdot \chi_2!}$. Consider the disjoint union graph $G_1 = H \uplus H$, which is disconnected, and satisfies $\chi(G_1) = \chi_1$ and $\mathcal{C}_{\chi_1}(G_1) = N_{\chi_1!^2 \chi_2!^2}$. By Corollary 3.3 applied with $k = \chi_2$ and $r = \chi_2! \cdot \chi_1!^2$, there exists a connected graph G_2 with $\chi(G) = \chi_2$ such that $\mathcal{C}_{\chi_2}(G_2) = N_{k! \cdot r} = N_{\chi_2! \cdot (\chi_2! \cdot \chi_1!^2)} = N_{\chi_1!^2 \chi_2!^2}$. \square

These results show that, in general, a graph cannot be uniquely reconstructed from its χ -coloring graph; however, exceptions exist. Our next result demonstrates that the 3-coloring graph of the 5-cycle is unique among all χ -coloring graphs.

Theorem 3.5. *If G is a graph satisfying $\mathcal{C}_{\chi(G)}(G) \cong \mathcal{C}_3(C_5)$, then $G \cong C_5$.*

Proof. Observe that $\mathcal{C} := \mathcal{C}_3(C_5)$ is the disjoint union of two copies of C_{15} . For any graph H , the number $|V(\mathcal{C}_k(H))|$ is divisible by $k!$. Since $|V(\mathcal{C}_{\chi(G)}(G))| = |V(\mathcal{C})| = 30$ is not divisible by $k!$ for $k \geq 4$, it follows that $\chi(G) \leq 3$. Moreover, since $|V(\mathcal{C})| = 30$ is not a power of 2, we have $\chi(G) > 2$. Thus, $\chi(G) = 3$.

First, we show that $|V(G)| \geq 5$. As G is *not* bipartite, G contains an odd cycle. If G contained a triangle, then \mathcal{C} would have at least $3! = 6$ connected components; indeed, a 3-coloring of a triangle is locked and cannot change within a connected component of \mathcal{C} . However, \mathcal{C} has only two components. Thus, G contains an ℓ -cycle where $\ell \geq 5$, so $|V(G)| \geq 5$.

Consider one of the 15-cycles in \mathcal{C} associated with recoloring vertices v_1, \dots, v_r . Let $\lambda_i \geq 2$ be the number of times vertex v_i changes color in this cycle, so that $(\lambda_1, \dots, \lambda_r)$ forms an integer partition of 15. We claim that $\lambda_i \geq 3$ for every i . If $\lambda_j = 2$ for some j , then vertex v_j is recolored (in this 15-cycle) between only two colors, say $1 \leftrightarrow 2$. Permuting these two colors with the third available color generates two *additional* 15-cycles in \mathcal{C} : one corresponding to v_j switching between $1 \leftrightarrow 3$, and the other corresponding to v_j switching between $2 \leftrightarrow 3$. Together with the original 15-cycle, this would force $|V(\mathcal{C})| \geq 3 \cdot 15 = 45$, a contradiction.

Combining $\lambda_i \geq 3$ for all i and $\lambda_1 + \dots + \lambda_r = 15$, we get $r \leq 5$. We claim that $r = 5$. If $r \leq 4$, then $|V(G)| \geq 5$ implies that there is a vertex $u \in V(G)$ whose color is fixed in the 15-cycle, say color 1. Changing its color to 2 or 3 results in two additional 15-cycles, implying $|V(\mathcal{C})| \geq 45$, a contradiction.

Thus, $r = 5$, which combined with $\lambda_i \geq 3$ for all i , forces $\lambda_i = 3$ for $1 \leq i \leq 5$. We claim that $|V(G)| = 5$. Indeed, any vertex $w \in V(G) \setminus \{v_1, \dots, v_5\}$ would have a fixed color in the 15-cycle, say color 1. Changing its color to 2 or 3 results in two additional 15-cycles, implying $|V(\mathcal{C})| \geq 45$, a contradiction. Therefore, $|V(G)| = 5$ and since G contains an ℓ -cycle with $\ell \geq 5$, we have $G \cong C_5$. \square

Later, Theorem 4.28 will show that $\mathcal{C}_3(C_5) \not\cong \mathcal{C}_k(G)$ for any graph G with $k > \chi(G)$. Combining this fact with Theorem 3.5, we conclude that C_5 is the only graph having a coloring graph isomorphic to $\mathcal{C}_3(C_5)$.

4. ANY COLORING GRAPH WITH SURPLUS COLORS HAS A UNIQUE BASE GRAPH

In this section, we establish a strong reconstruction result: if $\mathcal{C}_k(G) \cong \mathcal{C}_{k'}(G')$ where $k > \chi(G)$ and $k' \geq \chi(G')$, then $G' \cong G$ and $k' = k$. As shown in the previous section, the conclusion relies critically on $k > \chi(G)$, a condition that indicates surplus colors in at least one of the two graphs. The two main theorems in this section are Theorems 4.4 and 4.28. Both rely on the notion of a *link k -coloring* of a graph G with $k > \chi(G)$, a k -coloring in which each connected component of G uses the minimum necessary number of colors. We show that link k -colorings correspond to *abstract link vertices* of a graph \mathcal{C} . Theorem 4.4 shows that for any \mathcal{C} , there is at most one pair (G, k) with $k > \chi(G)$ such that $\mathcal{C} \cong \mathcal{C}_k(G)$. The proof strategy involves extracting information from link colorings and their corresponding abstract link vertices. The argument is constructive: Algorithm 1 reconstructs this unique graph by outputting its adjacency matrix.

Showing that $\mathcal{C}_k(G) \cong \mathcal{C}_{k'}(G')$ is impossible when $k > \chi(G)$ and $\chi' = \chi(G')$ (Theorem 4.28) requires several new techniques. The presence of abstract link vertices in $\mathcal{C} \cong \mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$ imposes specific constraints on how G can relate to a potential G' . The first step is to count the induced squares incident to each abstract link vertex in \mathcal{C} . These tallies provide adjacency information about the base graph vertices

that are changing color, and shows that any such G' must contain the connected components of G as vertex-disjoint induced subgraphs. Next, Algorithm 2 shows that at each abstract link vertex, this subgraph of G in G' must be colored according to the partition of the coloring in the surplus coloring case. Algorithm 3 then extracts from \mathcal{C} an auxiliary structure, termed *labeled link graph*, in a base-graph agnostic manner. Combined with structural properties of 3-, 4-, and 6-cycles in coloring graphs, these algorithms explain how different abstract link vertices in \mathcal{C} are interconnected. We establish that equivalent abstract link vertices (those in the same orbit under a certain group action) are connected by a predictable sequence of hypercubes, which correspond to edges in the labeled link graph. Finally, a counting argument leads to a numerical inconsistency. Interpreting the shared graph \mathcal{C} as $\mathcal{C}_{\chi'}(G')$ requires it to contain strictly more abstract link vertices than are present when interpreting it as $\mathcal{C}_k(G)$, contradicting the possibility that $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$. A flowchart at the end of the paper (Figure 12) illustrates the progression of results within this section.

4.1. Reconstructing the base graph via link vertices. This subsection proves Theorem 4.4: if an abstract coloring graph \mathcal{C} is isomorphic to $\mathcal{C}_k(G)$ for some graph G and an integer $k > \chi(G)$, then this pair (G, k) is unique and algorithmically recoverable from \mathcal{C} . The core idea is to identify abstract link vertices (Definition 4.2) within the graph \mathcal{C} . Algorithm 1 uses the structure around these special vertices both to identify them and to reconstruct the unique G and k . This algorithm relies solely on the structure of \mathcal{C} as an abstract graph. We first define the corresponding concept in the base graph, the link k -coloring (Definition 4.1), which corresponds to colorings where the image size within each connected component is minimal (equal to the component's chromatic number). Lemma 4.3 confirms that these concrete colorings map precisely to the abstract link vertices when $\mathcal{C} \cong \mathcal{C}_k(G)$ for $k > \chi(G)$. We begin with the formal definition of a link k -coloring.

Definition 4.1. Given a graph G with connected components H_1, \dots, H_d and an integer $k > \chi(G)$, a *link k -coloring* of G is a k -coloring $\alpha_G: V(G) \rightarrow [k]$ such that the image of $V(H_i)$ under α_G has size $\chi(H_i)$ for all $i \in [d]$.

Equivalently, a link k -coloring maximizes the number of common free colors for all vertices within each connected component.

We use Algorithm 1 to find vertices in an abstract graph \mathcal{C} that correspond to link k -colorings. The algorithm analyzes the structure of \mathcal{C} to identify a set of candidate vertices. When $\mathcal{C} \cong \mathcal{C}_k(G)$ with $k > \chi(G)$, this process succeeds and allows us to reconstruct the unique pair (G, k) . The vertices successfully identified by this algorithm are called *abstract link vertices*, formally defined in Definition 4.2.

Algorithm 1 (Reconstruction and Link Vertex Identification). Given a graph \mathcal{C} , identify a subset of vertices $A \subseteq V(\mathcal{C})$, a graph G , and an integer k as follows:

- (1) Let A be the set of vertices $\alpha \in V(\mathcal{C})$ where the neighborhood of α is a disjoint union of cliques. For each $\alpha \in A$, let r be the number of cliques in its neighborhood, and label them arbitrarily v_1, \dots, v_r .
- (2) Let n be the maximum number of cliques in the neighborhood of any vertex in A . Remove from A any vertex that does not attain this maximum (i.e., if $r < n$).
- (3) For each $\alpha \in A$, let t_1, \dots, t_n be the sizes of the n cliques in its neighborhood. Let m_α be the number of pairs $1 \leq i < j \leq n$ with strictly fewer than $t_i t_j$ squares spanning the i th and j th clique in the neighborhood of α . Let m be the maximum value of m_α over all $\alpha \in A$. Remove from A any vertex α for which $m_\alpha < m$.
- (4) For each $\alpha \in A$, let $M_\alpha \in \{0, 1\}^{n \times n}$ be the matrix such that $M_{ij} = 1$ if and only if there are strictly fewer than $t_i t_j$ squares spanning the i th and j th clique in the neighborhood of α .
- (5) For each $\alpha \in A$ and connected component H_ℓ induced by M_α , let f_ℓ be the maximum number of ways to select one vertex from each clique in the neighborhood of α that corresponds to a vertex in H_ℓ such that there is no square spanning the vertices corresponding to v_i, v_j for any i, j with $M_{i,j} = 1$. For each of these f_ℓ sets of $|V(H_\ell)|$ neighbors of α , label these edges with a distinct $c \in [f_\ell]$.
- (6) Let f be the maximum value of $\sum_\ell f_\ell$ over all $\alpha \in A$. Remove from A any α for which $\sum_\ell f_\ell < f$.
- (7) Let G be the graph with connected components H_1, \dots, H_d , as determined by an arbitrary adjacency matrix M_α from Step 4. Calculate $k = \frac{1}{d} \cdot \left(f + \sum_{\ell \in [d]} \chi(H_\ell) \right)$, where f is the maximum value from Step 6.
- (8) If $\mathcal{C} \not\cong \mathcal{C}_k(G)$, abort. Otherwise, return vertex set A , graph G , and integer k .

Definition 4.2. Given a graph \mathcal{C} , an *abstract link vertex* of \mathcal{C} is any vertex α belonging to the subset $A \subseteq V(\mathcal{C})$ returned by Algorithm 1. In other words, an abstract link vertex is one that survives Step 6, provided the algorithm successfully identifies a graph G and an integer k such that $\mathcal{C} \cong \mathcal{C}_k(G)$.

Lemma 4.3. For a graph G and $k > \chi(G)$, there is a natural bijection between the link k -colorings of G and the abstract link vertices of $\mathcal{C}_k(G)$.

Proof. We show that the natural bijection $\Phi: V(\mathcal{C}_k(G)) \rightarrow \{k\text{-colorings of } G\}$ restricts to a bijection between the abstract link vertices of $\mathcal{C}_k(G)$ and the link k -colorings of G whenever $k > \chi(G)$. It suffices to prove that the image of the abstract link vertices under Φ contains only link k -colorings and that every link k -coloring is achieved by some abstract link vertex. Recall that using the $\alpha \mapsto \alpha_G$ notation from Section 2, the symbol α represents an element of $V(\mathcal{C}_k(G))$ and $\alpha_G = \Phi(\alpha)$ is the corresponding k -coloring of base graph G .

Let α_G be a link k -coloring of G . Suppose G has d connected components H_1, \dots, H_d . We show that $\alpha = \Phi^{-1}(\alpha_G)$ remains in set A after all steps of Algorithm 1. The neighborhood of α in $\mathcal{C}_k(G)$ consists of the disjoint union of $n = |V(G)|$ cliques, because $k > \chi(G)$ implies that each vertex of G has at least one available color in a link k -coloring. Moreover, all vertices in $V(\mathcal{C})$ satisfy the condition in Step 1 for any coloring graph \mathcal{C} . Indeed, the edges of any clique in a coloring graph correspond to a single vertex changing color, and any coloring has a finite number of vertices with free colors, yielding a neighborhood of cliques. In particular, no vertex of \mathcal{C} can have a neighborhood with more than n cliques, so α survives Step 2.

Next, consider Step 3. For every edge $v_i v_j \in E(G)$, the corresponding cliques in the neighborhood of α have fewer than $t_i t_j$ squares. This is because v_i and v_j have at least one common free color in a link k -coloring, and a square corresponding to assigning this same color to both vertices is necessarily missing since $v_i v_j \in E(G)$. Conversely, for any $v_i v_j \notin E(G)$, the corresponding cliques have exactly $t_i t_j$ squares, as any combination of free colors for v_i and v_j produces a valid k -coloring. Hence, $m_\alpha = |E(G)|$ for this α . Additionally, no $\beta \in V(\mathcal{C})$ can have $m_\beta > |E(G)|$, because this would require some i, j with fewer than $t_i t_j$ squares but $v_i v_j \notin E(G)$, a contradiction. Therefore, α achieves the maximum m and survives Step 3.

Now consider Step 6. By definition of link k -coloring, for each $\ell \in [d]$, every vertex in $V(H_\ell)$ shares exactly $k - \chi(H_\ell)$ common free colors in α_G . Each of these common free colors is associated with $n_\ell = |V(H_\ell)|$ neighbors of α which pairwise have spanning squares if and only if the pair of corresponding vertices in G are non-adjacent. Thus, for each component H_ℓ , the value of f_ℓ calculated in Step 5 is exactly $f_\ell = k - \chi(H_\ell)$. Moreover, for any $\beta \in V(\mathcal{C}_k(G))$, a missing square between any two neighbors of β implies that the corresponding two vertices of G take on the same color moving from β to the respective neighbors. Thus, maximality of the definition of link k -colorings ensures that α survives Step 6 and is included in the set A returned by the algorithm.

Conversely, we show that any abstract link vertex α in $\mathcal{C}_k(G)$ corresponds to some link k -coloring α_G of G . For α to be in the final set A :

- Surviving Step 2 means that *every* vertex $v_i \in V(G)$ has at least one free color in α_G .
- Surviving Step 3 implies $m_\alpha = |E(G)|$. Hence, for every edge $v_i v_j \in E(G)$, the vertices v_i and v_j share at least one common free color in α_G .
- Surviving Step 6 requires $\sum_\ell f_\ell$ to be maximal. This corresponds precisely to colorings where each component H_ℓ uses exactly $\chi(H_\ell)$ colors, maximizing the number of common free colors within that component. By Definition 4.1, we deduce that α_G is a link k -coloring.

Therefore, Φ restricted to the set of abstract link vertices is a bijection onto the set of link k -colorings. \square

Figure 2 presents three non-link colorings that are rejected at different steps of Algorithm 1 and one link coloring that survives. The labels within braces next to a given vertex indicate its free available colors.

In the proof of Lemma 4.3, we saw that the vertices γ that survive Step 2 correspond to k -colorings in which each base graph vertex has a free color. Similarly, the vertices β that survive Step 3 represent those colorings in which every pair of adjacent base graph vertices share a common free color. All such colorings yield the same adjacency matrix in Step 4. This shared adjacency matrix is central to Theorem 4.4, which guarantees the uniqueness of the graph G and the value of k produced by the algorithm.

Theorem 4.4. For any graph G and $k > \chi(G)$, there is no graph G' and $k' > \chi(G')$ with $G \not\cong G'$ or $k \neq k'$ such that $\mathcal{C}_k(G) \cong \mathcal{C}_{k'}(G')$. Moreover, Algorithm 1 produces the unique (up to graph isomorphism) G and k .

Proof. Suppose that $C_k(G) \cong C_{k'}(G')$ for graphs G and G' where $k > \chi(G)$ and $k' > \chi(G')$. We show that $G \cong G'$ and $k = k'$. The isomorphism $C \cong C_k(G) \cong C_{k'}(G')$ means that Algorithm 1, run on C , identifies the *same* set A of abstract link vertices, regardless of whether we consider C as $C_k(G)$ or $C_{k'}(G')$. By Lemma 4.3, this set A is in bijection with the link k -colorings of G , and also in bijection with the link k' -colorings of G' .

Algorithm 1 (Step 2) identifies n as the maximum number of cliques in the neighborhood of any abstract link vertex. This number n corresponds to $|V(G)|$ when interpreting C as $C_k(G)$ and to $|V(G')|$ when interpreting C as $C_{k'}(G')$. Therefore, $|V(G)| = |V(G')| = n$. Algorithm 1 (Steps 3 and 4) derives an adjacency matrix M_α based on the pattern of missing squares between neighborhood cliques of abstract link vertices. Since this pattern is solely determined by the structure of C , it yields the same M_α (up to permutation), regardless of whether C is interpreted as $C_k(G)$ or $C_{k'}(G')$. As M_α represents the adjacency matrix of the base graph in both interpretations (because $k > \chi(G)$ and $k' > \chi(G')$), it follows that $G \cong G'$.

Furthermore, the algorithm calculates the maximum value $f = \sum_\ell f_\ell$ in Step 6. Using the formula from Step 7, we have $k = \frac{1}{d} \left(f + \sum_{\ell \in [d]} \chi(H_\ell) \right)$ and $k' = \frac{1}{d'} \left(f + \sum_{\ell \in [d']} \chi(H'_\ell) \right)$. Since $G \cong G'$, the connected components of the two graphs are isomorphic. After relabeling the component indices, we have $d = d'$ and $H_\ell \cong H'_\ell$ for each $\ell \in [d]$. As f is also derived uniquely from C , it follows that $k = k'$. \square

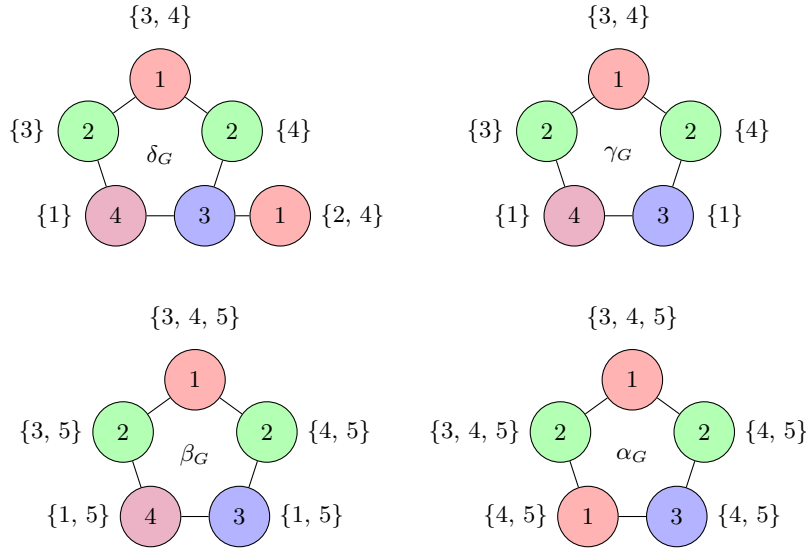


FIGURE 2. The vertex $\delta \in V(C_4(G))$ corresponding to coloring δ_G does not survive Step 2 of Algorithm 1 because the vertex colored 3 has no free color (free colors are indicated by colors within braces); $\gamma \in V(C_4(C_5))$ survives Step 2 but not Step 3 of Algorithm 1 because each vertex in γ_G has a free color but not all adjacent vertices share a common free color; $\beta \in V(C_5(C_5))$, survives Step 3 but not Step 6 because every pair of adjacent vertices in β_G has a common free color but there is only one common free color for all vertices; α_G is a link 5-coloring of C_5 such that $\alpha \in V(C_5(C_5))$ survives Algorithm 1 because colors 4 and 5 are free for each vertex and no other coloring can have more than $5 - \chi(C_5) = 2$ common free colors.

Remark 4.5. The adjacency matrix of G can be reconstructed not only from abstract link vertices in $V(C_k(G))$, but also from certain other vertices that we can completely characterize. Specifically, for any coloring in which every pair of adjacent vertices in G shares a common free color, the corresponding vertex in $C_k(G)$ will also yield the correct adjacency matrix. We refer to such colorings as *weak link k -colorings* of G . Unlike link k -colorings, weak link k -colorings do not require that *all vertices* within a given connected component share a common free color; they only require that *each pair* of adjacent vertices in G shares a common free color (which may differ between pairs). For example, the coloring β_G in Figure 2 is a weak

link 5-coloring, but not a link 5-coloring. The concept of link k -colorings, along with Algorithm 1, provides a systematic framework for identifying vertices from which the graph G can be uniquely reconstructed. In contrast, it remains unclear how the vertices corresponding to weak link k -colorings can be reliably identified using only the structure of the coloring graph.

In the following subsections, we will extract more information from the coloring graph. We remark that the definition of abstract link vertices applies to any abstract graph \mathcal{C} , including those of the form $\mathcal{C}_{\chi(G')}(G')$. This is true even though the concept of “link $\chi(G')$ -colorings” is not defined analogously to link k -coloring where $k > \chi(G)$. However, it is not yet clear whether Algorithm 1 can successfully run to completion on a graph $\mathcal{C}_{\chi(G')}(G')$ without aborting. For example, consider the 5-vertex graph G' formed by connecting a vertex of K_2 to a vertex of K_3 by a single edge. Its 3-coloring graph, $\mathcal{C} = \mathcal{C}_3(G')$, has 24 vertices: \mathcal{C} is the disjoint union of six paths of length 3 (that is, on four vertices). The degree-2 vertices in \mathcal{C} survive Step 6 of Algorithm 1, yielding the parameters $n = 2, m = 1, f = 1$ for a potential base graph. The resulting matrix M in Step 4 corresponds to $G = P_2$, and Step 7 then calculates $k = f + \chi(P_2) = 1 + 2 = 3$. However, the algorithm aborts in Step 8 because the input graph \mathcal{C} is *not* isomorphic to $\mathcal{C}_3(P_2) \cong C_6$.

The proof that $\mathcal{C}_{\chi(G')}(G')$ can *never* survive Algorithm 1 will follow from Theorem 4.28; for now, we observe that the properties of abstract link vertices already place structural constraints on G' . These constraints relate to the number and adjacency of vertices in G' that have free colors available at the corresponding coloring. In the previous example, consider an interior vertex α of one of the copies of P_4 in \mathcal{C} . Let β_1, β_2 be the two neighbors of α . The edges $\alpha\beta_1$ and $\alpha\beta_2$ in \mathcal{C} are neither part of a common clique nor part of an induced square. This necessarily represents a coloring in G' in which only two vertices, say u and v , have free colors, and all other vertices have their colors locked (that is, cannot be recolored). Furthermore, the lack of squares indicates that u and v must be adjacent in G' . In other words, if $\mathcal{C}_k(G')$ is isomorphic to the disjoint union of six copies of P_4 , then G' contains P_2 as a subgraph. This is related to the fact that the adjacency matrix produced by Step 4 corresponds to P_2 . Lemma 4.6 generalizes this observation.

Lemma 4.6. If $\mathcal{C} \cong \mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$ where $k > \chi(G)$ and $\chi' = \chi(G')$, then G' contains G as a subgraph. Moreover, if G consists of connected components H_1, \dots, H_d , then G' contains the connected components of G as vertex-disjoint induced subgraphs. That is, there exists an injection $\phi: V(G) \rightarrow V(G')$ such that for any $i \in [d]$ and $u, v \in H_i$, we have $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G')$.

Proof. By Theorem 4.4, we know that Algorithm 1 run on \mathcal{C} produces a unique (up to permutation) adjacency matrix M_α associated with any abstract link vertex. The isomorphism $\mathcal{C} \cong \mathcal{C}_{\chi'}(G')$ induces a mapping $\alpha \mapsto \alpha_{G'}$ so that each abstract link vertex $\alpha \in V(\mathcal{C})$ corresponds to a χ' -coloring $\alpha_{G'}$ of G' . The neighborhood structure around α in \mathcal{C} , specifically the cliques and the pattern of missing induced squares analyzed in Step 3 and Step 4 of Algorithm 1, implies that the corresponding χ' -coloring $\alpha_{G'}$ in G' has exactly $n = |V(G)|$ vertices with at least one free color. Furthermore, a pair of these vertices corresponding to missing induced squares in \mathcal{C} must be adjacent in G' . Since M_α is the adjacency matrix of G by Theorem 4.4, it follows that G is a subgraph of G' . We now make the inclusion $G \hookrightarrow G'$ explicit.

Label the n vertices in G as v_1, \dots, v_n , corresponding to the n cliques in the neighborhood of α in \mathcal{C} . Via the isomorphism $\mathcal{C} \cong \mathcal{C}_{\chi'}(G')$, each clique associated with v_j in the $\mathcal{C}_k(G)$ interpretation corresponds to changing the color of a unique vertex $\phi(v_j)$ in G' . This mapping ϕ is well-defined because each clique in the neighborhood of a vertex in $\mathcal{C}_{\chi'}(G')$ corresponds to the set of colorings obtained by changing the color of a single vertex in G' . In addition, the resulting map $\phi: V(G) \rightarrow V(G')$ is an injection.

As G' contains G as a subgraph, $\phi(u)\phi(v) \in E(G')$ for every edge $uv \in E(G)$. To show that each component $\phi(H_i)$ is an induced subgraph of G' , suppose for a contradiction that there exists an edge $\phi(u)\phi(v) \in E(G')$ with $uv \notin E(G)$ for some $u, v \in V(H_i)$. As $uv \notin E(G)$, by Theorem 4.4, every spanning square is present between the cliques in \mathcal{C} associated with base graph vertices u and v . Since u and v are in the same connected component H_i of G , there exists a path $u = w_1, w_2, \dots, w_{r-1}, w_r = v$ within H_i such that $w_j w_{j+1} \in E(G)$ for each $j \in [r-1]$. Thus, for each adjacent pair (w_j, w_{j+1}) on this path, there is at least one missing spanning square between their corresponding cliques in \mathcal{C} . The structure of missing squares along the path (when interpreting \mathcal{C} via G) implies that the vertices w_1, \dots, w_r must share at least one common free color in the link k -coloring in G , α_G , associated with α . Interpreting \mathcal{C} via G' , the same structural properties hold: squares are missing between cliques for $\phi(w_j)$ and $\phi(w_{j+1})$, while all squares are present between cliques for $\phi(w_1) = \phi(u)$ and $\phi(w_r) = \phi(v)$. By the required structure of the missing

spanning squares, there must be a common free color for all vertices $\phi(w_1), \dots, \phi(w_r)$. In particular, this implies that $\phi(u)$ and $\phi(v)$ have a common free color. Since all spanning squares exist between the cliques for $\phi(u)$ and $\phi(v)$, any combination of assigning a free color f_u to $\phi(u)$ and a free color f_v to $\phi(v)$ results in a valid coloring. Since they share a common free color, we can set $f_u = f_v$, resulting in a proper coloring of G' where $\phi(u)$ and $\phi(v)$ have the same color. This contradicts the assumption that $\phi(u)\phi(v) \in E(G')$. \square

Under the same hypothesis of Lemma 4.6, we will show that G' has strictly more vertices than G . We first recall the following folklore lemma [Wes01, Exercise 5.2.7].

Lemma 4.7. Given a proper k -coloring of a graph H with $\chi(H) = k$, for each $1 \leq i \leq k$, there exists a vertex v with color i such that v is adjacent to vertices of the other $k - 1$ colors.

Proof. Fix any color i , and let X_i be the set of vertices colored i . Suppose, for contradiction, that *every* $v \in X_i$ is missing at least one color in its neighborhood $N(v)$. That is, for each $v \in X_i$, there is some color $j \neq i$ not present in $N(v)$. We recolor v with j while still maintaining a proper coloring. By repeating this process for every $v \in X_i$, we eliminate color i entirely, resulting in a proper $(k - 1)$ -coloring of H . This contradicts the assumption that $\chi(H) = k$. Thus, there is at least one vertex $v_i \in X_i$ whose neighborhood includes all $k - 1$ other colors. \square

Lemma 4.8. If $\mathcal{C} \cong \mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$ where $k > \chi(G)$ and $\chi' = \chi(G')$, then $|V(G')| \geq |V(G)| + \chi'$.

Proof. Under the interpretation of \mathcal{C} as $\mathcal{C}_k(G)$, let $\alpha \in V(\mathcal{C})$ be an abstract link vertex, and let α_G be the corresponding link k -coloring of G . The neighborhood of α consists of $n = |V(G)|$ distinct cliques associated with vertices in G changing their current color at α_G to a free color. Under the interpretation of \mathcal{C} as $\mathcal{C}_{\chi'}(G')$, the same vertex α represents a proper χ' -coloring $\alpha_{G'}$ of G' . The same n cliques around α must arise from n distinct vertices in G' that can be recolored from their current color at $\alpha_{G'}$. By Lemma 4.7, there are at least χ' vertices of G' that are *locked* in the coloring $\alpha_{G'}$, that is, vertices in G' that cannot be recolored. Consequently, $|V(G')| \geq |V(G)| + \chi'$. \square

Since $\chi' > 0$, Lemma 4.8 has the following immediate consequence.

Corollary 4.9. Given a graph G and $k > \chi(G)$, there is no graph G' with $|V(G)| = |V(G')|$ such that $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi(G')}(G')$.

This corollary demonstrates that coloring graphs with surplus colors are complete graph invariants, when restricted to base graphs of the same order. Our main result Theorem 4.28 strengthens Corollary 4.9 by removing the hypothesis $|V(G)| = |V(G')|$. When G and G' have different orders, proving that $\mathcal{C}_k(G) \not\cong \mathcal{C}_{\chi(G')}(G')$ is highly non-trivial. The remainder of the paper is devoted to resolving this task by examining the structure of coloring graphs with surplus colors and identifying features absent in χ -coloring graphs.

The following subsections introduce techniques that go beyond the scope of Algorithm 1 to explore further structural properties that must hold if $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi(G')}(G')$ with $k > \chi(G)$. Ultimately, these properties will lead to a contradiction, proving that such an isomorphism is impossible.

4.2. Link vertices are associated with unique coloring partitions of a base (sub)graph. We now introduce Algorithm 2. Given $\mathcal{C} \cong \mathcal{C}_k(G)$ with $k > \chi(G)$ and an abstract link vertex α of \mathcal{C} , this algorithm labels certain edges incident to α in \mathcal{C} . This labeling corresponds to a unique *component-wise partition* that reflects the underlying coloring α_G of G . The resulting labeling is a key ingredient in Lemma 4.12. This important lemma establishes that if $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi(G')}(G')$, then the underlying coloring $\alpha_{G'}$ of G' corresponding to α has the same component-wise partition identified from G , when restricted to the induced copies of the components of G (see Lemma 4.6).

Definition 4.10. Given a k -coloring $\alpha_G: V(G) \rightarrow \{1, 2, \dots, k\}$ of graph G , the α_G -*partition* of $V(G)$ is $\{\alpha_G^{-1}(1), \dots, \alpha_G^{-1}(k)\}$. If G has d connected components H_1, \dots, H_d , the *component-wise partition* for α_G is $P(\alpha_G) = (P^1, \dots, P^d)$ where $P^i = \{\alpha_G^{-1}(1) \cap V(H_i), \dots, \alpha_G^{-1}(k) \cap V(H_i)\}$ for each $i \in [d]$. The *component-wise integer partition* for α_G is the tuple $\lambda(\alpha_G) = (\lambda^1, \dots, \lambda^d)$ where $\lambda_j^i = |P_j^i|$ for all $i \in [d], j \in [\ell(P^i)]$.

The component-wise partitions define equivalence classes of k -colorings of a graph G with connected components H_1, \dots, H_d . We describe this equivalence in terms of group actions. Let $S_k^d = S_k \times \dots \times S_k$ denote the d -fold product of the symmetric group S_k on k symbols. For $\sigma = (\sigma_1, \dots, \sigma_d) \in S_k^d$ and a

k -coloring α_G of G , define $\sigma \cdot \alpha_G$ to be a k -coloring by assigning the colors $(\sigma \cdot \alpha_G)(v) := \sigma_i(\alpha_G(v))$ for $v \in V(H_i)$. This defines a group action of S_k^d on the set of k -colorings of G . We say two k -colorings α_G and β_G are *equivalent* if they belong to the same orbit under this S_k^d action (that is, $\beta_G = \sigma \cdot \alpha_G$ for some $\sigma \in S_k^d$). Each orbit is associated with a unique component-wise partition P . Any k -coloring α_G consistent with P can serve as a representative for its equivalence class. However, non-equivalent k -colorings (belonging to different orbits) may have the same component-wise *integer* partition. Figure 3 presents such an example.

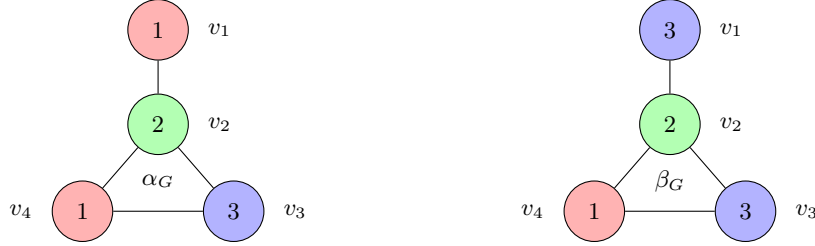


FIGURE 3. Non-equivalent link 4-colorings α_G and β_G of the paw graph G have distinct component-wise partitions, $P(\alpha_G) = (\{\{v_1, v_4\}, \{v_2\}, \{v_3\}\})$ and $P(\beta_G) = (\{\{v_1, v_3\}, \{v_2\}, \{v_4\}\})$. However, while non-equivalent, these colorings have the same component-wise integer partition, $\lambda(\alpha_G) = \lambda(\beta_G) = (211)$.

The following algorithm extends the color labels to edges of certain hypercubes incident to a given abstract link vertex α of a graph \mathcal{C} . Subsequent lemmas establish the uniqueness of this labeling (up to color permutations). Specifically, Lemma 4.11 shows that when $\mathcal{C} \cong \mathcal{C}_k(G)$ for $k > \chi(G)$, Algorithm 2 correctly identifies the unique component-wise partition $P(\alpha_G)$ associated with the link k -coloring α_G .

Algorithm 2 (Partition Extraction). Given a graph \mathcal{C} and abstract link vertex $\alpha \in V(\mathcal{C})$, identify a list of partitions P for α_G as follows:

- (1) Run Algorithm 1 to obtain G with connected components H_1, \dots, H_d and $k > \chi(G)$ such that $\mathcal{C} \cong \mathcal{C}_k(G)$, or abort if reconstruction fails.
- (2) For every $i \in [d]$ and free color $c \in [f_i]$ for f_i as in Step 5 of Algorithm 1, let $V_{i,c} \subseteq V(\mathcal{C})$ be the set of $|V(H_i)|$ neighbors of α corresponding to vertices in H_i being recolored to color c from α .
- (3) For every partition $P^i \vdash V_{i,c}$ with $\ell(P^i) = \chi(H_i)$, determine whether to retain P^i as follows.
- (4) For each $j \in [\chi(H_i)]$, check whether the neighbors of α corresponding to P_j^i extend to a hypercube in \mathcal{C} , and reject P^i and go back to Step 3 if any does not.
- (5) For α_j^i antipodal to α on the hypercube associated with P_j^i , first check whether α_j^i is an abstract link vertex according to Algorithm 1. If not, reject P^i and return to Step 3. Otherwise, check whether all neighbors of α_j^i in the hypercube correspond to a common free color at α_j^i . If not, reject P^i and return to Step 3. Otherwise, retain P^i , assign color $f_i + j$ as the color of each $v \in P_j^i$ at α_G , and proceed to find a partition for the next connected component.
- (6) Return the successfully identified component-wise partition $P = (P^1, \dots, P^d)$.

The following lemma states that Algorithm 2 correctly identifies the unique component-wise partition for a link k -coloring when run on the surplus coloring graph $\mathcal{C} \cong \mathcal{C}_k(G)$ with $k > \chi(G)$. The result holds regardless of how the base graph vertices $v_1, \dots, v_{|V(G)|}$ are initially labeled in Algorithm 1 (provided the labeling is consistent with the adjacency matrix). Figure 4 below explains how Algorithm 2 unfolds for the 4-coloring graph of the paw graph from Figure 3. Note that Algorithm 1 could have interchanged the v_3 and v_4 labels due to the automorphism of G .

Lemma 4.11. Let $\mathcal{C} \cong \mathcal{C}_k(G)$ for $k > \chi(G)$. Let $\alpha \in V(\mathcal{C})$ be an abstract link vertex corresponding to a link k -coloring α_G of G . Then Algorithm 2, run with input \mathcal{C} and vertex α , returns the unique component-wise partition for α_G . In particular, for each connected component H_i of G where $i \in [d]$, exactly one candidate partition P^i will satisfy the conditions in Steps 4 and 5.

Proof. Let $P(\alpha_G) = (P^1, \dots, P^d)$ be the component-wise partition for link k -coloring α_G . Fix $i \in [d], j \in [\chi(H_i)]$ and any color c among the $f_i = k - \chi(H_i)$ common free colors for vertices in H_i from coloring α_G . Let $(\alpha_j^i)_G$ be the link k -coloring obtained from α_G by recoloring every vertex in P_j^i to color c , which has the same component-wise partition $P(\alpha_G)$. By Lemma 4.3, the vertex $\alpha_j^i = \Phi^{-1}((\alpha_j^i)_G)$ in \mathcal{C} must be an abstract link vertex. Moreover, $\alpha = \Phi^{-1}(\alpha_G)$ and α_j^i are antipodal vertices of a $|P_j^i|$ -dimensional hypercube in \mathcal{C} . This is because the underlying link k -coloring $(\alpha_j^i)_G$ differs from α_G by a partial transposition taking α_G 's j th part of the i th connected component and recoloring it c . Furthermore, the neighbors of α_j^i within this hypercube correspond to changing the vertices in P_j^i from color c back to their original common color in α_G . Thus, this choice of $P = P(\alpha_G)$ satisfies all the checks in Algorithm 2.

Now, consider a different candidate partition $\tilde{P} = (\tilde{P}^1, \dots, \tilde{P}^d)$ for α_G . If $\tilde{P}^i \neq P^i$ for some $i \in [d]$, then at least one part $\tilde{P}_j^i \in \tilde{P}^i$ contains two vertices v and w with $\alpha_G(v) \neq \alpha_G(w)$. When Algorithm 2 considers the part \tilde{P}_j^i , there are two possibilities. First, neighbors of α corresponding to changing vertices in \tilde{P}_j^i may not survive Step 4. Otherwise, these vertices will extend to a hypercube whose antipodal vertex $\tilde{\alpha}$ corresponds to a coloring $\tilde{\alpha}_G$ where v and w have color c . Upon running Algorithm 1 at $\tilde{\alpha}$, the path of recolorings returning to α involves changing v back to $\alpha_G(v)$ and w back to $\alpha_G(w)$. Since these colors are different, the neighbors of $\tilde{\alpha}$ on the return path do not correspond to a single common free color, failing Step 5. Therefore, $P(\alpha_G)$ is the unique component-wise partition for α_G returned by the algorithm. \square

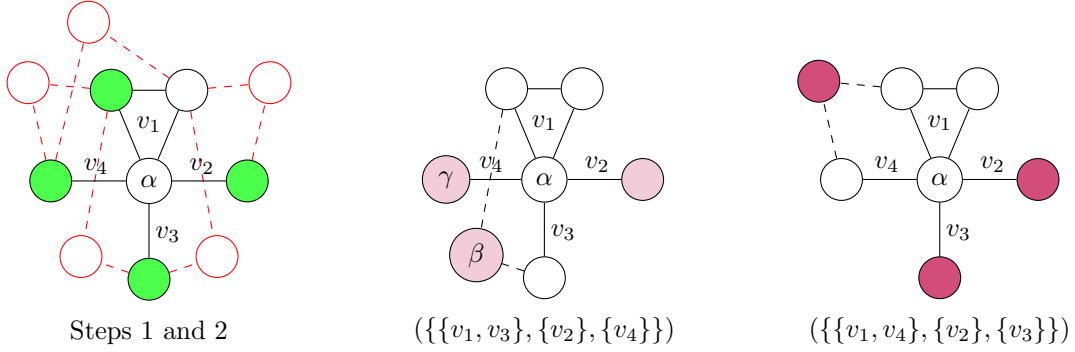


FIGURE 4. Algorithm 2 for the link k -coloring $\alpha_G = 1231$ in $\mathcal{C}_4(G)$ from Figure 3. First, Step 1 associates all incident edges at $\alpha \in V(\mathcal{C}_4(G))$ with vertex labels from the base graph consistent with the reconstructed graph G . Step 2 identifies all of the green highlighted vertices in $\mathcal{C}_4(G)$ as changing to a common free color (in this case color 4), as indicated by the sequence of missing squares between them. We then start testing partitions of length 3. The partition $(\{\{v_2, v_3\}, \{v_1\}, \{v_4\}\})$ fails Step 4 as there is no hypercube extended from the green vertices labeled v_2 and v_3 . The partition $(\{\{v_1, v_3\}, \{v_2\}, \{v_4\}\})$ survives this step but fails Step 5 as the vertex γ corresponding to $\gamma_G = 1234$ in the second picture is not an abstract link vertex. Moreover, while β with $\beta_G = 4241$ is an abstract link vertex, it fails Step 5 as its return trip to $\alpha = 1231$ does not use a common free color (in this case color 3). The correct partition $(\{\{v_1, v_4\}, \{v_2\}, \{v_3\}\})$ survives every step in Algorithm 2 as the colored vertices in the third picture are all abstract link vertices that have a common free color returning to α .

When applied to an abstract link vertex α in a coloring graph $\mathcal{C}_k(G)$ with $k > \chi(G)$, Algorithm 2 assigns a specific proper coloring $\alpha_G: V(G) \rightarrow [k]$. Moreover, by providing free color labels and identifying the correct partition, Algorithm 2 consistently assigns a proper coloring to all hypercube-adjacent abstract link vertices α_j^i . However, this consistency is only guaranteed within this local neighborhood. Running the algorithm on an abstract link vertex β that is not hypercube-adjacent to α may produce a labeling that is inconsistent with the labeling assigned to α and its hypercube neighborhood. To remedy this limitation, Algorithm 3 in Section 4.3 enhances Algorithm 2 through a more sophisticated labeling process that maintains consistency across equivalent abstract link vertices.

Assuming the hypothetical isomorphism $\mathcal{C} \cong \mathcal{C}_k(G) \cong \mathcal{C}_{\chi(G')}(G')$, we make the following observation. The labels for common free colors identified by Step 5 in Algorithm 1 that reconstructs G also correspond to common free colors for the relevant vertices in G' . In remainder of the paper, we will often refer to a specific hypothesis.

Hypothesis 1. *Suppose $\mathcal{C} \cong \mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$ where $k > \chi(G)$ and $\chi' = \chi(G')$. This pair of isomorphisms, $\mathcal{C} \cong \mathcal{C}_k(G)$ and $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$, will be fixed, as will the underlying colorings of G and G' corresponding to each vertex in \mathcal{C} . Assume H_1, \dots, H_d are the connected components of G . If $\alpha \in V(\mathcal{C})$ is an abstract link vertex, then α_G and $\alpha_{G'}$ will be the corresponding link k -coloring of G and χ' -coloring of G' , respectively. By Lemma 4.6, G' contains subgraphs H'_1, \dots, H'_d such that $H'_i \cong H_i$ for each $i \in [d]$. We also recall the embedding $\phi: V(G) \rightarrow V(G')$ from Lemma 4.6.*

Next, we show that Algorithm 2 successfully extracts the unique vertex partition P^i for each induced component H'_i in G' at the same abstract link vertex of \mathcal{C} .

Lemma 4.12. *Assume Hypothesis 1. For each $i \in [d]$, $\alpha_{G'}$ restricted to $V(H'_i)$ uses exactly $\chi(H_i)$ colors. Furthermore, there exists an injection $\psi_i: [k] \rightarrow [\chi']$ such that $\alpha_{G'}(\phi(v)) = \psi_i(\alpha_G(v))$ for each $v \in V(H_i)$. That is, the partition of $V(H'_i)$ induced by $\alpha_{G'}$ is equivalent (via ϕ and ψ_i) to the partition P^i of $V(H_i)$ induced by α_G .*

Proof. Lemma 4.11 ensures that Algorithm 2, run on \mathcal{C} and α , correctly identifies the unique component-wise $P(\alpha_G) = (P^1, \dots, P^d)$ associated with the link k -coloring α_G . Since the algorithm relies only on the structure of \mathcal{C} , it must identify the same partitions when interpreting \mathcal{C} as $\mathcal{C}_{\chi'}(G')$. We will confirm that the identified partition P^i corresponds to a valid proper coloring of $V(H'_i)$ represented by the coloring $\alpha_{G'}$.

By Lemma 4.6, each H'_i is an induced subgraph of G' . Therefore, the adjacency relations within H'_i are consistent with H_i , and the partition P^i defines independent sets *within* H'_i . There is only one potential issue: the edges between $V(H'_i)$ and $V(G') \setminus V(H'_i)$ might conflict with the coloring pattern dictated by P^i .

However, the structural analysis performed by Algorithm 2 works equally well for the G' interpretation. Suppose α, α' are antipodal abstract link vertices on a hypercube associated with changing an *independent* part $T \in P^i$ from color c to color \bar{c} . This structure in \mathcal{C} implies that the corresponding χ' -colorings $\alpha_{G'}$ and $\bar{\alpha}_{G'}$ differ exactly on the vertices of $\phi(T)$, where $\alpha_{G'}(\phi(v)) = c$ and $\bar{\alpha}_{G'}(\phi(v)) = \bar{c}$ for all $v \in T$. The same analysis holds for all such hypercube transitions identified by Algorithm 2. Crucially, Algorithm 2 analyzes each component H_i (and thus H'_i) independently based on its internal structure and common free colors. It ensures the partition P_i reflects the coloring within H'_i . It does not, however, determine relationships between the color sets used for different components H'_i and H'_j .

Therefore, for each induced subgraph H'_i of G' , the identified partition P^i accurately reflects the color classes within $V(H'_i)$ under the coloring $\alpha_{G'}$. This is formalized by defining, for each component H_i , a map $\psi_i: [k] \rightarrow [\chi']$ that translates the colors used by α_G on H_i to those used by $\alpha_{G'}$ on H'_i . Concretely, ψ_i maps the color $\alpha_G(v)$ to the color $\alpha_{G'}(\phi(v))$ for each $v \in V(H_i)$. The choice of ψ_i is not unique, because it only needs to be prescribed on the set of $\chi(H_i)$ colors used by the link k -coloring α_G . By the preceding discussion, ψ_i is injective and the relation $\alpha_{G'}(\phi(v)) = \psi_i(\alpha_G(v))$ holds for each $v \in H_i$ by definition. \square

Lemma 4.12 guarantees that the internal coloring structure of each subgraph H'_i in G' matches the corresponding H_i in G (up to color permutation) at any abstract link vertex α . However, it does not constrain the relationship between the actual sets of colors used. For instance, the set of $\chi(H_i)$ colors used in H'_i might partially or fully overlap with the set of $\chi(H_j)$ colors used in H'_j . By analyzing $\mathcal{C}_{\chi'}(G')$ at an abstract link vertex, Algorithm 2 determines the partition within each H'_i consistently. However, we cannot guarantee that the color labels we have chosen for individual H'_i patch together in a consistent way to yield a proper coloring $\alpha_{G'}$ of G' . The subtlety arises because there may be additional edges in G' , including edges between $u \in V(H'_i)$ and $v \in V(H'_j)$. See Figure 5 for an example of this phenomenon.

Another key subtlety arises when applying Lemma 4.12 in the $\mathcal{C}_{\chi'}(G')$ context. The graph G' may contain multiple copies of the subgraph G (or its components H_i). Merely identifying a partition corresponding to H'_i at an abstract link vertex α does *not* guarantee we are looking at the same copy of H_i within G' when we move to a different abstract link vertex β (see Figure 6). However, the algorithm introduced in the next subsection helps resolve this issue by analyzing how different link vertices with the same vertex partition are connected within \mathcal{C} . This will effectively fix the specific subgraph of G' under consideration, which allows

reasoning about the interaction between this fixed subgraph and the rest of G' . We achieve our goal by introducing a new structure, termed *labeled link graph*. The vertices in this new graph are link vertices in a given equivalence class, with an edge between each pair of equivalent link vertices that are antipodal vertices of a hypercube in the underlying coloring graph. As demonstrated in the following subsection, analyzing the labeled link graph associated with a link vertex α in \mathcal{C} removes the ambiguity about which G subgraph of G' is being considered.

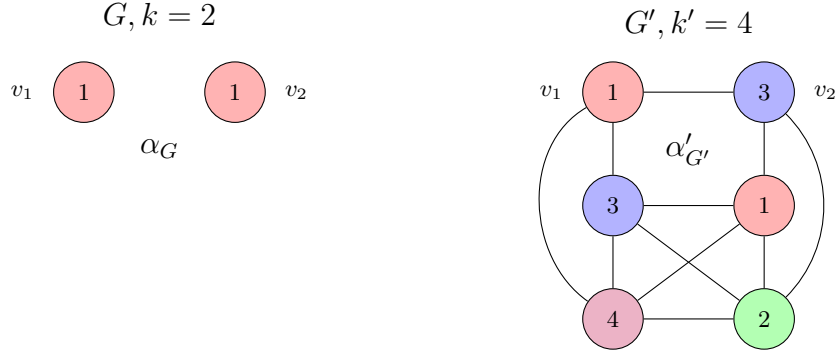


FIGURE 5. A 2-coloring α_G of G consisting of two isolated vertices, where $k > \chi$ and a 3-coloring $\alpha'_{G'}$ of G' where $k' = \chi'$. In both α_G and $\alpha'_{G'}$, only v_1 and v_2 have exactly one free color, and they can be recolored independently. The neighborhood structure around abstract link vertex $\alpha \in \mathcal{C}_k(G)$ and $\alpha' \in \mathcal{C}_{\chi'}(G')$ are identical. Applying Lemma 4.12, we identify the trivial partitions $\{v_1\}$ for $H_1 \cong H'_1$ and $\{v_2\}$ for $H_2 \cong H'_2$. This reflects $\alpha'_{G'}(v_1) = 1$ and $\alpha'_{G'}(v_2) = 3$ in G' . However, this local analysis does not reveal the edge $v_1 v_2$ in G' , so we cannot infer whether v_1 and v_2 can take the same color. We note that G and G' will not have equivalent coloring graphs; this example merely illustrates a subtlety between locally identical vertices in $\mathcal{C}_k(G)$ and $\mathcal{C}_{\chi'}(G')$.

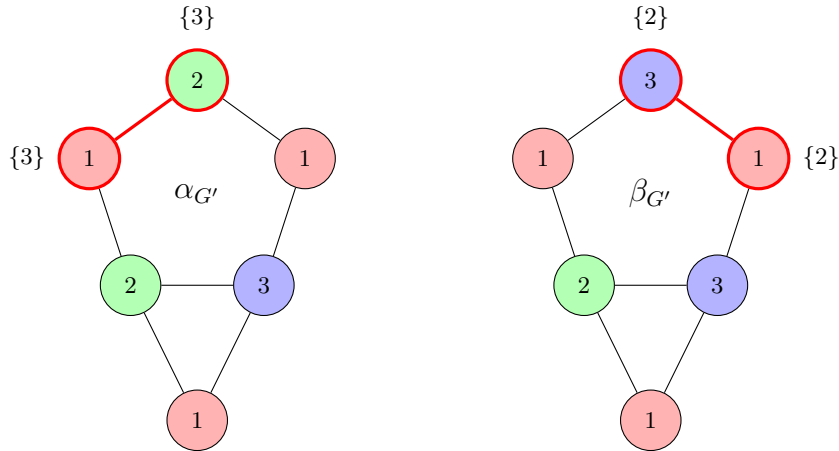


FIGURE 6. A 6-vertex graph G' and two colorings $\alpha_{G'}$ and $\beta_{G'}$ which are adjacent in $\mathcal{C}_{\chi(G')}(G')$. When run on any vertex in $\mathcal{C}_3(G')$, Algorithm 1 will reconstruct the graph $G = P_2$ and $k = 3 > \chi(G)$, as exactly two adjacent vertices in G' have a common free color and all other vertices are locked. However, the corresponding subgraph $G = P_2$ in G' changes depending on the underlying proper χ' -coloring of G' .

4.3. Equivalent link vertices and labeled link graphs. Building on Algorithm 2, we now extend the labeling process for a coloring graph with surplus colors. For $\mathcal{C} \cong \mathcal{C}_k(G)$ where $k > \chi(G)$, we aim for a consistent labeling of the hypercube structures connecting equivalent link vertices. We first establish a series of lemmas demonstrating how equivalent link k -colorings are connected within \mathcal{C} .

Let α_G be a link k -coloring of graph G with connected components H_1, \dots, H_d where $k > \chi(G)$. Let $i \in [d]$, and let σ be a transposition in the symmetric group S_k swapping two colors in $[k]$.

Lemma 4.13. Define the coloring $\sigma_i(\alpha_G)$ by applying the transposition σ only in H_i by setting $\sigma_i(\alpha_G) = \sigma(\alpha_G(v))$ for $v \in V(H_i)$ and $\sigma_i(\alpha_G) = \alpha_G(v)$ for $v \notin V(H_i)$. Then the corresponding vertices α and $\sigma_i(\alpha)$ of $\mathcal{C}_k(G)$ are in the same connected component.

Proof. Suppose without loss of generality that G is connected (so $d = 1$) and $\sigma = (12)$. The argument extends to disconnected graphs because σ does not affect vertices outside H_i .

Case 1. Neither 1 nor 2 is in the image of α_G .

In this case, $\alpha_G = (12)\alpha_G$, so the corresponding vertices are identical and thus trivially connected.

Case 2. Exactly one of 1 or 2 is in the image of α_G .

Then α and $(12)\alpha$ are antipodal vertices on a cube of dimension $\max\{|\alpha_G^{-1}(1)|, |\alpha_G^{-1}(2)|\}$, because the vertices colored 1 in α_G (or 2, if none are 1 in α_G) form the dimensions of a hypercube representing their colors independently changing from 1 to 2 (or 2 to 1, if none are 1 in α_G) moving from α to $(12)\alpha$.

Case 3. Both 1 and 2 are in the image of α_G .

Since $k > \chi(G)$, there must be at least one color not used by α_G . Without loss of generality, assume that color 3 is not in the image of α_G . By the same reasoning from Case 2, the sequence of link colorings $(\alpha, (13)\alpha, (12)(13)\alpha, (23)(12)(13)\alpha)$ are pairwise antipodal vertices connected via hypercubes, and hence α and $(23)(12)(13)\alpha = (12)\alpha$ are connected. \square

Let $V_1 = \alpha_G^{-1}(1)$ and $V_2 = \alpha_G^{-1}(2)$. In Case 3 of Lemma 4.13, the path of connectivity is established by traversing hypercubes. We describe this transformation explicitly by labeling the path of hypercubes connecting α and $(12)\alpha$ as follows:

$$(4.1) \quad \alpha \xrightarrow{1V_13} (13)\alpha \xrightarrow{2V_21} (12)(13)\alpha \xrightarrow{3V_12} (23)(12)(13)\alpha = (12)\alpha$$

where the notation $c_1 V c_2$ indicates that every vertex in part V changes its color from c_1 to c_2 .

We refer to the equivalence relation introduced after Definition 4.10. Lemma 4.13 extends to show connectivity between *any* two equivalent link k -colorings in $\mathcal{C}_k(G)$. This follows because any element in the group $S_k^d = S_k \times \dots \times S_k$ (acting on the set of link k -colorings) can be generated by a sequence of transpositions within components, and Lemma 4.13 handles connectivity for each transposition.

Corollary 4.14. If α_G and β_G are equivalent link k -colorings of G with $k > \chi(G)$, then the corresponding vertices in $\mathcal{C}_k(G)$ are in the same connected component.

The proof of Lemma 4.13 also reveals how equivalent colorings differing on partial transpositions are connected via hypercubes of a certain dimension. Every nontrivial transposition of α is connected to α by either one hypercube (Case 2), or a chain of three hypercubes (Case 3). Furthermore, in Case 3 (where α_G uses 1 and 2, and we assume 3 is free), there exists a second path of three hypercubes connecting α to $(12)\alpha$:

$$(4.2) \quad \alpha \xrightarrow{2V_23} (23)\alpha \xrightarrow{1V_12} (12)(23)\alpha \xrightarrow{3V_12} (13)(12)(23)\alpha = (12)\alpha.$$

The two paths (4.1) and (4.2) are internally disjoint; their union creates a *6-cycle of cubes*. Figure 1 illustrates how $\mathcal{C}_3(P_3)$ is a 6-cycle of cubes of alternating dimensions 1 and 2. This holds more generally for $\mathcal{C}_3(K_{n_1, n_2})$, noting that $P_3 \cong K_{1,2}$, as formalized by the following corollary and illustrated in Figure 7.

Corollary 4.15. Let α_G be any link k -coloring of G with components H_1, \dots, H_d and $k > \chi(G)$. Suppose α_G has component-wise integer partition vector $\lambda(\alpha_G) = (\lambda^1, \dots, \lambda^d)$ where $\lambda^i \vdash \chi(H_i)$. Then for each $i \in [d]$ and any two indices $1 \leq j_1 < j_2 \leq \chi(H_i)$, the coloring graph $\mathcal{C}_k(G)$ contains $\mathcal{C}_3(K_{\lambda_{j_1}^i, \lambda_{j_2}^i})$ as a subgraph.

Figure 7 shows an example of a 6-cycle of cubes, where for a base graph G the parts defined by $V_1 = \{v_4, v_5, v_6\}$ and $V_2 = \{v_1, v_3\}$ cycle through colors 1, 3, and 4. As $|V_1| = 3$ and $|V_2| = 2$ the resulting subgraph in $\mathcal{C}_4(G)$ is isomorphic to $\mathcal{C}_3(K_{3,2})$. The six abstract link vertices are named *links* because they serve as common vertices shared by a family of cycles within $\mathcal{C}_3(K_{|V_1|, |V_2|})$. Each cycle in this family is composed

of three paths of length $|V_1|$ and three paths of length $|V_2|$, with each path traversing one of the six cubes, for a total length of $3(|V_1| + |V_2|)$. This entire structure $\mathcal{C}_3(K_{|V_1|,|V_2|})$, visualized as a 6-cycle of cubes, is an induced subgraph of $\mathcal{C}_k(G)$. Moreover, every 6-cycle of cubes is also “induced” from a hypercube perspective. This means the six link vertices are only adjacent to each other (via hypercubes) as dictated by the 6-cycle structure, with each vertex connecting to exactly two others.

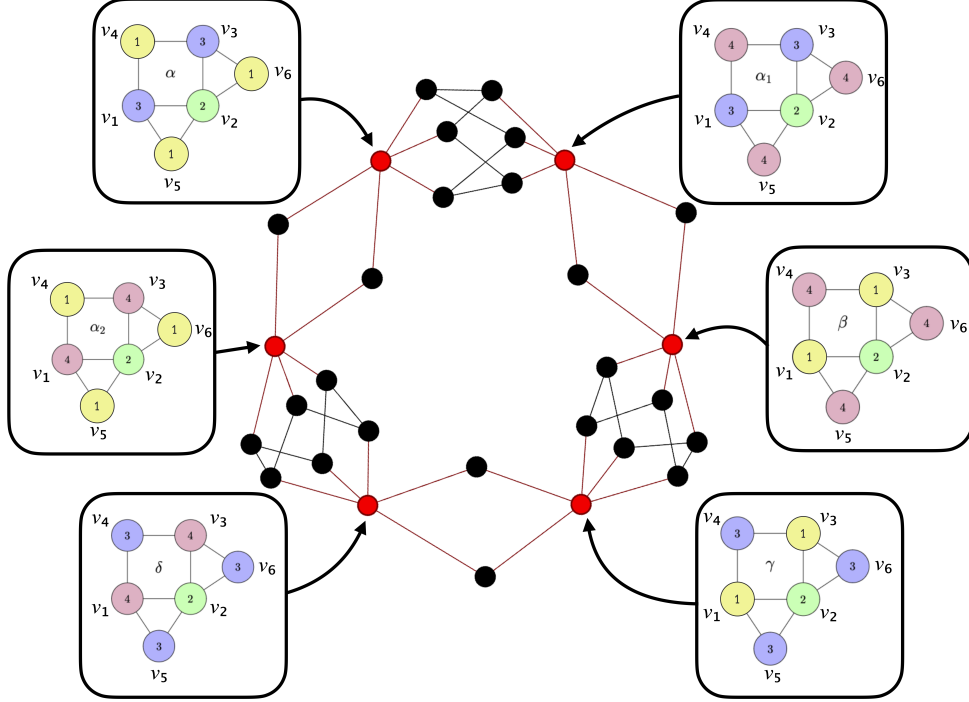


FIGURE 7. The 6-cycle of cubes defined by abstract link vertex α , free color 4, and parts $V_1 = \{v_4, v_5, v_6\}$ and $V_2 = \{v_1, v_3\}$. The 6 red vertices are the 6 abstract link vertices that participate in the 6-cycle. All black vertices are the parts V_1 or V_2 independently changing colors to reach the next abstract link vertex. In our notation, this 6-cycle of cubes is described as $\alpha \xrightarrow{1V_1 4} \alpha_1 \xrightarrow{3V_2 1} \beta \xrightarrow{4V_1 3} \gamma \xrightarrow{1V_2 4} \delta \xrightarrow{3V_1 1} \alpha_2 \xrightarrow{4V_2 3} \alpha$.

The results in this subsection so far pertain to structures in coloring graphs with surplus colors. Algorithm 3 describes a procedure to identify hypercubes associated with some abstract link vertex α to propagate consistent cube labels, even without assuming surplus colors. The algorithm begins with an abstract link vertex α and the color and vertex labels for its base graph as provided by Algorithms 1 and 2. By iteratively rerunning Algorithm 2 and analyzing 6-cycles of cubes, Algorithm 3 ensures that color and vertex labels are consistent across neighboring link vertices. This process continues until the entire labeled link graph associated with α is generated and labeled. In Lemma 4.19, we prove that labeling decisions made by Algorithm 3 are uniquely determined for surplus coloring graphs (up to the choice of a bijection $\alpha \mapsto \alpha_G$).

Recall that the group S_k^d acts on the set of proper k -colorings of G . If α_G is a link k -coloring, then $\sigma \cdot \alpha_G$ is also a link k -coloring. Hence, the action descends to the set of link k -colorings. Given a link k -coloring α_G in G , we denote by $[\alpha_G]$ the equivalence class of α_G (that is, the orbit of α_G under this group action). Suppose \mathcal{C} is an abstract coloring graph arising from a graph G with surplus colors; that is, $\mathcal{C} \cong \mathcal{C}_k(G)$ where $k > \chi(G)$. Using Lemma 4.3, there is a bijection between abstract link vertices α in \mathcal{C} and link k -colorings α_G in G . We define $[\alpha]$ as the set of vertices in \mathcal{C} that represents the colorings in $[\alpha_G]$ under this bijection.

Definition 4.16. Suppose we are given $\mathcal{C} \cong \mathcal{C}_k(G)$ along with the map $\alpha \mapsto \alpha_G$. Assume $k > \chi(G)$ and we have an abstract link vertex α corresponding to link k -coloring α_G with component-wise partition $P(\alpha_G) = (P^1, \dots, P^d)$. Define $\mathcal{L}_{[\alpha]}$ as an edge-labeled graph where:

- (1) Vertices of $\mathcal{L}_{[\alpha]}$: abstract link vertices β in $[\alpha]$, that is, β_G is equivalent to α_G .

- (2) Edges of $\mathcal{L}_{[\alpha]}$: β and γ are adjacent if β_G and γ_G differ in the color of exactly one part P_j^i .
- (3) Edge labels: If β_G and γ_G differ on the part P_j^i with $\beta_G(P_j^i) = c_1$ and $\gamma_G(P_j^i) = c_2$, we label the edge $\beta\gamma$ by:

$$\beta \xleftrightarrow{c_1 P_j^i c_2} \gamma.$$

We refer to $\mathcal{L}_{[\alpha]}$ as a *labeled link graph* of $\mathcal{C}_k(G)$ corresponding to an equivalence class $[\alpha]$.

The object $\mathcal{L}_{[\alpha]}$ is not intrinsic to the base graph G . There is a subtle dependence on the choice of the bijection $\alpha \mapsto \alpha_G$ that concretely realizes a vertex α in $V(\mathcal{C})$ as a function $\alpha_G: V(G) \rightarrow [k]$.

Lemma 4.17. The graph $\mathcal{L}_{[\alpha]}$, disregarding its edge labels, is isomorphic to $\mathcal{C}_k\left(\bigsqcup_{i \in [d]} K_{\chi(H_i)}\right)$.

A general fact [BFH⁺16, Lemma 1] states that the coloring graph of the disjoint union, $\mathcal{C}_k\left(\bigsqcup_{i \in [d]} G_i\right)$, is isomorphic to a Cartesian product of individual coloring graphs, $\square_{i \in [d]} \mathcal{C}_k(G_i)$.

Proof. A link k -coloring α_G (and any $\beta_G \in [\alpha_G]$) uses a specific set of $\chi(H_i)$ distinct colors for each connected component H_i of G . These $\chi(H_i)$ colors define the parts $P_1^i, \dots, P_{\chi(H_i)}^i$ of the component-wise partition P^i for H_i . In $\mathcal{L}_{[\alpha]}$, an edge corresponds to changing the color assigned to one such part, say P_j^i , to a new color, while the colors of all other parts P_t^s (for $(s, t) \neq (i, j)$) remain unchanged.

Consider the graph $K_{\chi(H_i)}$. Its vertices can be identified with the abstract “slots” for the $\chi(H_i)$ colors used in component H_i . A k -coloring of $K_{\chi(H_i)}$ assigns one of k colors to each of these $\chi(H_i)$ slots such that all slots receive distinct colors. An edge in $\mathcal{C}_k(K_{\chi(H_i)})$ connects two such colorings if they differ in the color assigned to exactly one slot.

This is precisely the structure of transitions within each component H_i as reflected in $\mathcal{L}_{[\alpha]}$: changing the color of a part P_j^i (a “slot”) to a new color (from the k available, distinct from colors of other parts P_m^i in H_i) corresponds to an edge in $\mathcal{C}_k(K_{\chi(H_i)})$. Since these changes can be made independently for each component H_i of G to move between colorings in $[\alpha_G]$, the overall structure of $\mathcal{L}_{[\alpha]}$ (unlabeled) is the Cartesian product of these individual coloring graphs, $\square_{i \in [d]} \mathcal{C}_k(K_{\chi(H_i)})$. This is isomorphic to the k -coloring graph of the disjoint union of these complete graphs, $\mathcal{C}_k\left(\bigsqcup_{i \in [d]} K_{\chi(H_i)}\right)$. \square

This structure implies that all of the 6-cycle of cubes that connect equivalent link k -colorings in $\mathcal{C}_k(G)$ become 6-cycles in the labeled link graph. Identifying (and ultimately labeling) these 6-cycles is a key step in constructing the labeled link graph. The following lemma states that if two consecutive edges in the 6-cycle are specified, then there is a unique continuation of the path that completes this 6-cycle. This uniqueness holds as long as the two edges are not also part of a 3- or 4-cycle. This condition is satisfied if the labeling of these edges is of the form $\alpha_1 \xleftrightarrow{c_3 P_{j_1}^i c_1} \alpha \xleftrightarrow{c_2 P_{j_2}^i c_3} \alpha_2$, where, from α , two different parts in the same connected component are sent to the same free common color c_3 .

Lemma 4.18. Suppose $\alpha_1, \alpha_2 \in [\alpha]$ are two vertices satisfying the following adjacency condition in $\mathcal{L}_{[\alpha]}$:

$$\alpha_1 \xleftrightarrow{c_3 P_{j_1}^i c_1} \alpha \xleftrightarrow{c_2 P_{j_2}^i c_3} \alpha_2.$$

Then there is a unique shortest path from α_1 to α_2 in $\mathcal{L}_{[\alpha]}$ that avoids α , which has length 4 given by:

$$\alpha_1 \xleftrightarrow{c_2 P_{j_2}^i c_1} \beta \xleftrightarrow{c_3 P_{j_1}^i c_2} \gamma \xleftrightarrow{c_1 P_{j_2}^i c_3} \delta \xleftrightarrow{c_2 P_{j_1}^i c_1} \alpha_2$$

for some vertices β, γ, δ .

Proof. Let t denote the length of the shortest α -avoiding path from α_1 to α_2 . We claim that $t = 4$. Suppose, to the contrary, that $t \leq 3$. If $t = 3$, then together with α , we have an induced 5-cycle in $\mathcal{L}_{[\alpha]}$. This is a contradiction, because C_5 is not an induced subgraph of any coloring graph [BFH⁺16, Corollary 12]. We may now assume $t \leq 2$. Since α_1 and α_2 are not adjacent, $t \neq 1$. Consequently, $t = 2$ and there exists a vertex $\zeta \neq \alpha$ between α_1 and α_2 in $\mathcal{L}_{[\alpha]}$:

$$\alpha_1 \xleftrightarrow{?} \zeta \xleftrightarrow{?} \alpha_2.$$

Now, $(\alpha_1)_G$ and $(\alpha_2)_G$ differ exactly on the two parts $P_{j_1}^i$ and $P_{j_2}^i$. By inspection, no such ζ can exist, as it would require for both $P_{j_1}^i$ and $P_{j_2}^i$ to be colored c_3 , which would be a contradiction as we could have fewer parts than the chromatic number. Thus, $t = 4$. The following α -avoiding path of length 4 from α_1 to α_2 certainly exists:

$$(4.3) \quad \alpha_1 \xleftarrow{c_2 P_{j_2}^i c_1} \beta \xleftarrow{c_3 P_{j_1}^i c_2} \gamma \xleftarrow{c_1 P_{j_2}^i c_3} \delta \xleftarrow{c_2 P_{j_1}^i c_1} \alpha_2$$

Next, we show that (4.3) is the unique such path of length 4.

Consider an arbitrary α -avoiding path of length 4 from α_1 to α_2 . When combined with the edges (α, α_1) and (α_2, α) , we obtain an induced 6-cycle $(\alpha, \alpha_1, \beta, \gamma, \delta, \alpha_2, \alpha)$ in $\mathcal{L}_{[\alpha]}$. By [ABFR18], an induced 6-cycle in a coloring graph arises in one of two situations: (a) two vertices alternately swapping between three colors, or (b) three vertices independently swapping between two colors. Case (b) always occurs as part of a 3-cube. Our setup involves two parts $P_{j_1}^i, P_{j_2}^i$, and specifies two colors for each of these parts (colors c_1 and c_3 for $P_{j_1}^i$ and colors c_2 and c_3 for $P_{j_2}^i$). If we had case (b), the 3-cube allows all color changes independently, so we would have a vertex where both $P_{j_1}^i$ and $P_{j_2}^i$ are colored c_3 , which would result in a proper coloring with the number of colors fewer than the chromatic number, a contradiction. Therefore, we must have case (a). The intermediate vertices of the 6-cycle are uniquely determined by the two involved parts and three colors. Thus, the remaining vertices β, γ, δ in any induced 6-cycle formed this way are unique, proving that (4.3) is the only shortest α -avoiding path of length 4 from α_1 to α_2 . \square

For an abstract link vertex $\alpha \in V(\mathcal{C})$, a modified use of Algorithm 2 can be applied to find all equivalent link vertices, which are precisely the vertices of $\mathcal{L}_{[\alpha]}$. After identifying the partition $P(\alpha_G)$ of α_G using Algorithm 2, it is clear that changing every vertex in any one part P_j^i to any common free color results in an equivalent link vertex, β . This process can be repeated at β to discover additional equivalent link vertices. Given the connectivity of the labeled link graph, conducting this search in breadth-first order guarantees that we will find all vertices in $[\alpha]$.

The subtlety in this approach is that although Algorithm 2 works equally well on both α and β , the exact labeling of the partitions is not necessarily consistent. Even with only one connected component, there is a potential mismatch. Suppose $P(\alpha_G) = P^1$ with the partition $P^1 = P_1^1 \uplus P_2^1 \uplus P_3^1$, where β_G is obtained by recoloring vertices in P_1^1 . There is no guarantee that Algorithm 2 run on β will label P_2^1 as P_2^1 again; it might instead be labeled P_3^1 .

The following algorithm, Algorithm 3, addresses this subtlety by not only identifying all of the vertices in $\mathcal{L}_{[\alpha]}$ but also consistently labeling the parts and colors (and thus the edges in $\mathcal{L}_{[\alpha]}$). Given an initial abstract link vertex α , Algorithms 1 and 2 are run to obtain a canonical labeling of vertices, colors, and the partition in Steps 1 and 2. A breadth-first search of all vertices in $\mathcal{L}_{[\alpha]}$ is then initialized in Steps 3 and 4. In these steps, every edge incident to α in $\mathcal{L}_{[\alpha]}$ is labeled using the canonical labeling from the previous steps.

The final Step 5 runs the breadth-first search until all vertices are found, while consistently labeling all edges of $\mathcal{L}_{[\alpha]}$. Given a complete set of labeled edges incident to a vertex γ in $\mathcal{L}_{[\alpha]}$, including the edge $\gamma\beta$, we analyze certain cycles to label all incident edges to β consistently. There are four such cases, which are depicted in Figure 8 and described in the ensuing algorithm. In the algorithm, the breadth-first ordering guarantees that when labeling the edges incident to some β , there is at least one predecessor γ (i.e., a neighbor of β that is closer to α) whose incident edges have already been labeled.

Algorithm 3 (Labeled Link Graph Identification). Given a graph \mathcal{C} and an abstract link vertex $\alpha \in V(\mathcal{C})$, identify $[\alpha]$ and construct its labeled link graph $\mathcal{L}_{[\alpha]}$ as follows:

- (1) Run Algorithm 1 to obtain G (or abort) with components H_1, \dots, H_d , labels on incident edges indicating color-changing vertices and the common free color palette $[f_i]$ for each $v \in H_i$ from α .
- (2) Run Algorithm 2 to assign canonical color labels $\{f_i + 1, \dots, f_i + \chi(H_i)\}$ to the colors used in α for vertices within each component H_i , consistent with its component-wise partition P_α .
- (3) Initialize the vertex set $V = \{\alpha\}$ and the edge set $E = \emptyset$ for $\mathcal{L}_{[\alpha]}$. Initialize an empty queue Q .
- (4) Iterate for each $i \in [d]$, each color $c \in [f_i]$, and each index $1 \leq j \leq \chi(H_i)$: Let $c_j = f_i + j$ denote the color of P_j^i at α assigned by Algorithm 2. Let β be the abstract link vertex corresponding to

recoloring part j with c from α . Add β to V and Q , and $\alpha\beta$ to E with label $\alpha \xleftarrow{c_j P_j^i c} \beta$.

- (5) While Q is nonempty: remove the next element β from queue Q and find a predecessor element γ in $V \setminus Q$ such that $\beta\gamma \in E$. Let $c_j P_j^i c$ be the existing label of this edge $\gamma\beta$ so that c_j is the color of P_j^i at γ and c is the color of P_j^i at β . For each $1 \leq \ell \leq d$, let \mathcal{F}^ℓ be the common free color palette at γ for H_ℓ ; note that $c \in \mathcal{F}^i$ and $c_j \notin \mathcal{F}^i$. We label all currently unlabeled incident edges to β (and possibly other edges) by analyzing incident edges to γ as follows:

- (a) **Iterate through incident edges to γ where parts in H_i change to same free color c :**

For each $1 \leq k \leq \chi(H_i)$ with $k \neq j$: use existing labels on edges incident to γ to identify the unique δ such that $\gamma \xleftrightarrow{c_k P_k^i c} \delta$. By exploring hypercube adjacent neighbors, and using Algorithm 2 to confirm that each vertex we consider is in $[\alpha]$, find the shortest sequence of vertices in $[\alpha] \setminus \{\gamma\}$ from β to δ where two consecutive vertices are antipodal points of hypercubes of alternating dimensions $|P_j^i|$ and $|P_k^i|$. By Lemma 4.18, this path is $\beta, \beta_1, \beta_2, \beta_3, \delta$. If not already in V , add $\beta_1, \beta_2, \beta_3$ (in this order) to V and Q . Add the following edges to E with labels

$$\gamma \xleftrightarrow{c_j P_j^i c} \beta \xleftrightarrow{c_k P_k^i c_j} \beta_1 \xleftrightarrow{c P_j^i c_k} \beta_2 \xleftrightarrow{c_j P_j^i c} \beta_3 \xleftrightarrow{c_k P_k^i c_j} \delta \xleftrightarrow{c P_k^i c_k} \gamma.$$

- (b) **Iterate through incident edges to γ where other parts in H_i change to different free color:**

For each $1 \leq k \leq \chi(H_i)$ with $k \neq j$ and $\tilde{c} \in \mathcal{F}^i \setminus \{c\}$: use existing labels on edges incident to γ to identify the unique δ such that $\gamma \xleftrightarrow{c_k P_k^i \tilde{c}} \delta$. As this color change is independent of recoloring $c_j P_j^i c$ on the edge $\gamma\beta$, there is a unique vertex $\beta_1 \in [\alpha]$ hypercube adjacent to both β and δ . If not already in V , add β_1 to V and Q , and add the following edges to E with labels

$$\gamma \xleftrightarrow{c_j P_j^i c} \beta \xleftrightarrow{c_k P_k^i \tilde{c}} \beta_1 \xleftrightarrow{c P_j^i c_j} \delta \xleftrightarrow{\tilde{c} P_k^i c_k} \gamma.$$

- (c) **Iterate through incident edges to γ where P_j^i changes to different free color:**

For each index $\tilde{c} \in \mathcal{F}^i \setminus \{c\}$: use existing labels on edges incident to γ to identify the unique δ such that $\gamma \xleftrightarrow{c_j P_j^i \tilde{c}} \delta$. This abstract link vertex δ is necessarily hypercube adjacent to β , so we add the edge $\delta \xleftrightarrow{\tilde{c} P_j^i c} \beta$ to E .

- (d) **Iterate through incident edges to γ where parts not in H_i change to a free color:**

For each $\ell \in [d] \setminus \{i\}$ and $\tilde{c} \in \mathcal{F}^\ell$: use existing labels on edges incident to γ to identify the unique δ such that $\gamma \xleftrightarrow{c_s P_k^\ell \tilde{c}} \delta$. As this color change is independent of recoloring $c_j P_j^i c$ on the edge $\gamma\beta$, there is a unique vertex $\beta_1 \in [\alpha]$ hypercube adjacent to both β and δ . If not already in V , add β_1 to V and Q , and add the following edges to E with labels

$$\gamma \xleftrightarrow{c_j P_j^i c} \beta \xleftrightarrow{c_s P_k^\ell \tilde{c}} \beta_1 \xleftrightarrow{c P_j^i c_j} \delta \xleftrightarrow{\tilde{c} P_k^\ell c_s} \gamma.$$

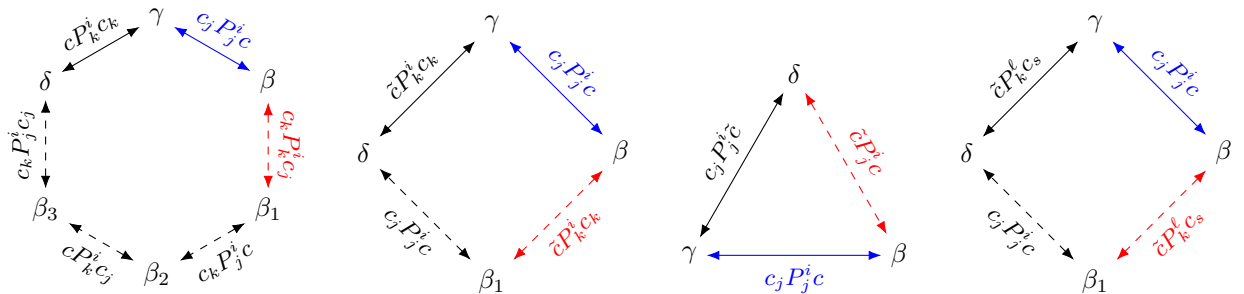


FIGURE 8. Illustration of Algorithm 3, Step 5, parts (a)-(d). The blue arrow indicates the edge from β to its predecessor γ . Solid edges are already labeled; dashed edges are in the process of being labeled (although some may have been previously labeled). Processing the red dashed edge $\beta \leftrightarrow \beta_1$ at each step eventually exhausts all incident edges to β .

As with our previous algorithms, we first show that the labeling produced by this algorithm is unique in some sense with respect to a fixed labeling of $\mathcal{C}_k(G)$ in the case when $k > \chi(G)$. Subsequently, we will show that it implies something structural about a version of the labeled link graph in the $\mathcal{C}_{\chi'}(G')$, $\chi' = \chi(G')$ interpretation under Hypothesis 1.

In Definition 4.16, the edge labeling in $\mathcal{L}_{[\alpha]}$ is uniquely determined because we implicitly fix a bijection $\alpha \mapsto \alpha_G$. Since we want Algorithm 3 to apply to an abstract coloring graph \mathcal{C} , there could be multiple labelings of $\mathcal{L}_{[\alpha]}$, depending on the choice of bijection $\alpha \mapsto \alpha_G$. Algorithms 1 and 2 fix such a mapping, and Algorithm 3 labels the labeled link graph consistent with this choice in its Step 1 and Step 2.

Lemma 4.19. Let $\mathcal{C} \cong \mathcal{C}_k(G)$ with $k > \chi(G)$ and let α be an abstract link vertex. Then Algorithm 3, when run on \mathcal{C} and α , exactly identifies $\mathcal{L}_{[\alpha]}$, where the edge labeling in $\mathcal{L}_{[\alpha]}$ is consistent with the labeling of \mathcal{C} produced by Algorithm 1 in Step 1.

Proof. First, Algorithm 3 finds all vertices of $\mathcal{L}_{[\alpha]}$, because all equivalent link k -colorings of G are connected by a sequence of hypercubes by Corollaries 4.14 and 4.15. Second, Algorithm 3 correctly identifies each abstract link vertex as being equivalent to α by the correctness of Algorithm 2. It remains to show that Algorithm 3 consistently labels all edges in $\mathcal{L}_{[\alpha]}$.

Every vertex in $\mathcal{L}_{[\alpha]}$ has degree $\sum_{\ell=1}^d \chi(H_\ell)(k - \chi(H_\ell))$. Indeed, for each connected component H_ℓ , any of its $\chi(H_\ell)$ parts can change to any of the $k - \chi(H_\ell)$ free colors. Step 5 finds and labels all of these edges, as (a) labels $\chi(H_i) - 1$ edges; (b) labels $(\chi(H_i) - 1)(k - \chi(H_i) - 1)$ edges; (c) labels $(k - \chi(H_i) - 1)$ edges; (d) labels $\sum_{\ell \neq i} \chi(H_\ell)(k - \chi(H_\ell))$ edges. These edges are all distinct. Summing these counts, and including the edge that connects to the predecessor, gives the desired total degree.

Moreover, we claim that these edge labels are uniquely determined and hence correct. For Step 5(a), we consistently label every 6-cycle by Lemma 4.18. For Step 5(b), we are recoloring two parts in the same connected component to different colors. There is a unique fourth vertex where both color changes happen, completing the induced square. Therefore, we can consistently label these color changes. Similar logic applies to Step 5(d), since color changes on parts in two distinct components H_i and H_ℓ ($\ell \neq i$) are independent. Lastly, for Step 5(c), if two edges of a 3-cycle are already labeled (representing a single part changing to two distinct free colors), we can uniquely label the third edge.

Finally, this labeling is consistent across all vertices β in $[\alpha]$ because Algorithm 3 relies on a breadth-first search (BFS) in Step 5. Starting with the canonical labeling at α , the BFS ensures that when any vertex β is processed, a neighboring vertex (its predecessor) has already been fully and consistently labeled, providing a valid reference. To be exact, in parts (b) and (d), only β_1 could be added to Q , and we are guaranteed a predecessor β as we are currently processing it. In part (a), we add $\beta_1, \beta_2, \beta_3$ to the queue in this order; since β_1 is adjacent to β (which is being processed), each vertex has a predecessor when dequeued. \square

We have shown that, given a labeling of a link k -coloring and all of its hypercube neighbors, we can uniquely construct the entire labeled link graph. Shifting to Hypothesis 1, where $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$, we aim to understand the corresponding proper χ' -colorings of G' and their transformations across this labeled link graph structure. By Lemma 4.12, every abstract link vertex of $\mathcal{C}_k(G)$ defines an analogous partition on a subgraph of G' . However, the precise color changes throughout the entire labeled link graph in the G' context are not yet clear. In particular, it is not yet apparent whether the subgraph of G' undergoing color changes remains fixed across all vertices of the labeled link graph, or whether more than k colors are involved. The following definition introduces $\mathcal{L}'_{[\alpha]}$, an alternative version of the labeled link graph, whose edge labels record the color changes of the underlying proper χ' -colorings in G' .

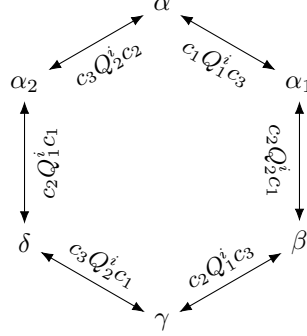
Each labeled edge in $\mathcal{L}_{[\alpha]}$ indicates that there is a hypercube between the two link vertices in \mathcal{C} . By Lemma 4.12, the same hypercube structure implies that the underlying colorings of G' also differ in the color of exactly one part. To keep track of this information, we create a new labeled graph.

Definition 4.20. Assume Hypothesis 1. Given an abstract link vertex α in \mathcal{C} , we define $\mathcal{L}'_{[\alpha]}$ as follows.

The graph $\mathcal{L}'_{[\alpha]}$ shares the same vertex set and (unlabeled) edge set with $\mathcal{L}_{[\alpha]}$. For each edge $\beta \xrightarrow{c_1 P_j^i c_2} \gamma$ in $\mathcal{L}_{[\alpha]}$, if $\beta_{G'}$ and $\gamma_{G'}$ differ in the color on part Q^i , we label the edge $\beta \xrightarrow{c_1 Q^i c_2} \gamma$ accordingly in $\mathcal{L}'_{[\alpha]}$.

Our next goal is to show that the edge labels in $\mathcal{L}'_{[\alpha]}$ have a natural correspondence to the edge labels in $\mathcal{L}_{[\alpha]}$. Identifying 3-, 4-, and 6-cycles play a crucial role for labeling $\mathcal{L}_{[\alpha]}$ via Algorithm 3. The following two lemmas assert that our interpretation of these cycles in $\mathcal{L}'_{[\alpha]}$ is the natural one. Lemma 4.21 shows that the 6-cycles (which are not part of a 3-cube) in $\mathcal{L}'_{[\alpha]}$ correspond to two fixed parts cycling between three colors. Lemma 4.22 states that 4-cycles in $\mathcal{L}'_{[\alpha]}$ arise from two fixed parts independently changing colors, while 3-cycles involve one fixed part changing among three colors.

Lemma 4.21. Assume Hypothesis 1. Let $\alpha, \alpha_1, \alpha_2 \in V(\mathcal{L}'_{[\alpha]})$ be vertices with the edge $\alpha\alpha_1$ labeled $c_1Q_1^ic_3$ and the edge $\alpha\alpha_2$ labeled $c_2Q_2^ic_3$. If α_1 and α_2 are non-adjacent, and α is their *only* common neighbor, then $\mathcal{L}'_{[\alpha]}$ has a unique 6-cycle containing the path $\alpha_1\alpha\alpha_2$, and its edge labels are given by:



Proof. The vertices $\alpha, \alpha_1, \alpha_2$ in $\mathcal{L}'_{[\alpha]}$ have corresponding elements in $\mathcal{L}_{[\alpha]}$. By Lemma 4.18 applied to $\mathcal{L}_{[\alpha]}$, the path $\alpha_1\alpha\alpha_2$ extends uniquely to an induced 6-cycle $(\alpha, \alpha_1, \beta, \gamma, \delta, \alpha_2, \alpha)$. For every vertex in the 6-cycle in $\mathcal{L}_{[\alpha]}$, the two outgoing incident edges in the 6-cycle represent changes to the same free color. This property continues to hold in the labeling of this 6-cycle in $\mathcal{L}'_{[\alpha]}$, as common free colors can be identified in the coloring graph. In particular, a part changing to a free color in $\alpha_1 \rightarrow \beta$ agrees with the free color $\alpha_1 \rightarrow \alpha$, which is c_1 . Figure 9 displays this partial information as $\alpha_1 \xrightarrow{?Wc_1} \beta$ for some subset W of vertices of G' . We can similarly deduce the partial information about edge $\alpha_2 \xrightarrow{?Zc_2} \delta$. By the structure of 6-cycle of cubes in $\mathcal{L}_{[\alpha]}$, the dimension of the cubes is alternating. Thus, $|Z| = |Q_1^i|$ and $W = |Q_2^i|$.

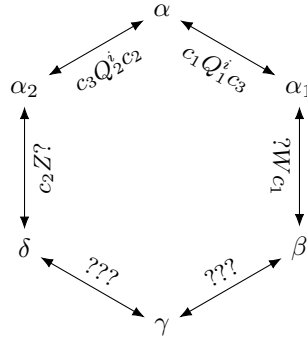


FIGURE 9. Partial information

We claim that $W = Q_2^i$. Suppose, to the contrary, that $W \neq Q_2^i$. Since $|W| = |Q_2^i|$, it follows that

$$(4.4) \quad |W \cap Q_2^i| < |Q_2^i| \quad \text{and} \quad Q_2^i \setminus W \neq \emptyset.$$

At vertex β , we have $\beta_{G'}(Q_1^i) = c_3$, $\beta_{G'}(Q_2^i \setminus W) = c_2$, and $\beta_{G'}(W) = c_1$. For the cycle to reach α_2 , the colors of these three distinct vertex sets (Q_1^i , $Q_2^i \setminus W$, and W) must all eventually change along the path segment $\beta \rightarrow \gamma \rightarrow \delta \rightarrow \alpha_2$. Specifically, over these three edges $\beta\gamma$, $\gamma\delta$ and $\delta\alpha_2$, there must be exactly one edge where vertices colored c_1 (those in W) change, one where vertices colored c_2 (those in $Q_2^i \setminus W$) change, and one where vertices colored c_3 (those in Q_1^i) change. Since the edge $\delta\alpha_2$ is already labeled by taking

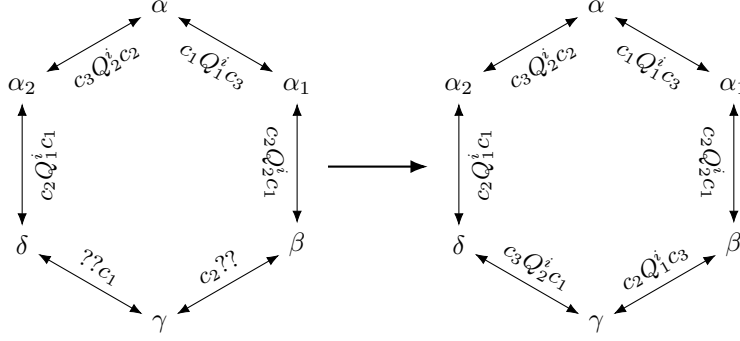


FIGURE 10. Identifying the final two edges in the 6-cycle

vertices colored c_2 to a new color, the vertices in $Q_2^i \setminus W$ must change on this edge back to c_3 . So, we may write $Z = (Q_2^i \setminus W) \uplus Z'$ for some set Z' (allowing for the possibility that $Z' = \emptyset$).

Next, observe that the shortest path between abstract link vertices α and δ in the actual coloring graph \mathcal{C} is $|Q_1^i| + |Q_2^i|$. There cannot be a shorter path as α_G and δ_G differ colors on exactly $|Q_1^i| + |Q_2^i|$ vertices, as these edges are correctly labeled as changing parts two distinct parts of dimension $|Q_1^i|$ and $|Q_2^i|$ in $\mathcal{L}_{[\alpha]}$. However, with our assumption $W \neq Q_2^i$, we construct a shorter path from α to δ_2 . Let $\tilde{Q}_2^i = W \cap Q_2^i$. Starting at α , consider the sequence of color changes given by $c_2\tilde{Q}_2^i c_3$ and then $c_3 Z' c_2$ to reach δ . Note that $|\tilde{Q}_2^i| = |W \cap Q_2^i| < |Q_2^i|$ and $|Z'| < |Z| = |Q_1^i|$ using $Q_2^i \setminus W \neq \emptyset$ from (4.4). We produced a path from α to δ of length $|Z'| + |\tilde{Q}_2^i| < |Q_1^i| + |Q_2^i|$, a contradiction.

Therefore, $W = Q_2^i$, and the edge $\alpha\beta$ is labeled $c_2 Q_2^i c_1$. By identical reasoning, we conclude $Z = Q_1^i$ and the edge $\alpha_2\delta$ is labeled $c_1 Q_1^i c_2$. We only have two remaining edges to identify as in Figure 10 (left). Both Q_1^i and Q_2^i must return to their original color as we go around the cycle. As the $\beta\gamma$ and $\gamma\delta$ edges already have one of the colors fixed in their edge label, there is only one possible way to label these two edges $\delta\gamma$ and $\gamma\beta$; see Figure 10 (right). \square

Lemma 4.22. Assume Hypothesis 1. Let $\alpha, \alpha_1, \alpha_2 \in V(\mathcal{L}'_{[\alpha]})$ be distinct vertices such that α is adjacent to both α_1 and α_2 .

- (1) If $\alpha\alpha_1$ has label $c_1 Q_1^i c_3$ and $\alpha\alpha_2$ has label $c_2 Q_2^i c_4$, where c_1, c_2, c_3 , and c_4 are all distinct, and Q_1^i and Q_2^i are disjoint sets, then there is a unique induced 4-cycle in $\mathcal{L}'_{[\alpha]}$ that contains these two edges and its labels are given by:

$$\alpha \xleftarrow{c_1 Q_1^i c_3} \alpha_1 \xleftarrow{c_2 Q_2^i c_4} \beta \xleftarrow{c_3 Q_1^i c_1} \alpha_2 \xleftarrow{c_4 Q_2^i c_2} \alpha.$$

- (2) If $\alpha\alpha_1$ has label $c_1 Q_1^i c_3$ and $\alpha\alpha_2$ has label $c_1 Q_1^i c_4$, where c_1, c_3 , and c_4 are all distinct, then there is a unique 3-cycle in $\mathcal{L}'_{[\alpha]}$ that contains these two edges and its labels are given by:

$$\alpha \xleftarrow{c_1 Q_1^i c_3} \alpha_1 \xleftarrow{c_3 Q_1^i c_4} \alpha_2 \xleftarrow{c_4 Q_1^i c_1} \alpha.$$

- (3) If $\alpha\alpha_1$ has label $c_1 Q_1^i c_3$ and $\alpha\alpha_2$ has label $c_2 Q_2^j c_4$ for $i \neq j$, where c_1 and c_3 , and c_2 and c_4 are necessarily pairwise distinct, and Q_1^i and Q_2^j are disjoint sets, then there is a unique induced 4-cycle in $\mathcal{L}'_{[\alpha]}$ that contains these two edges and its labels are given by:

$$\alpha \xleftarrow{c_1 Q_1^i c_3} \alpha_1 \xleftarrow{c_2 Q_2^j c_4} \beta \xleftarrow{c_3 Q_1^i c_1} \alpha_2 \xleftarrow{c_4 Q_2^j c_2} \alpha.$$

Proof. We prove each statement independently below.

- (1) We are given $\alpha_{G'}(Q_1^i) = c_1, \alpha_{G'}(Q_2^i) = c_2$, with edges $\alpha \xleftarrow{c_1 Q_1^i c_3} \alpha_1$ and $\alpha \xleftarrow{c_2 Q_2^i c_4} \alpha_2$. Thus, $(\alpha_1)_{G'}(Q_1^i) = c_3, (\alpha_1)_{G'}(Q_2^i) = c_2$ and $(\alpha_2)_{G'}(Q_1^i) = c_1, (\alpha_2)_{G'}(Q_2^i) = c_4$. Since Q_1^i, Q_2^i are disjoint

parts and c_1, c_2, c_3, c_4 are distinct, the recoloring operations $Q_1^i: c_1 \rightarrow c_3$ and $Q_2^j: c_2 \rightarrow c_4$ are independent. Let β be the coloring where $\beta_{G'}(Q_1^i) = c_3$ and $\beta_{G'}(Q_2^j) = c_4$ (other parts as in α).

This β is proper as $c_3 \neq c_4$. The edges $\alpha_1 \xleftrightarrow{c_2 Q_2^j c_4} \beta$ and $\beta \xleftrightarrow{c_3 Q_1^i c_1} \alpha_2$ complete the stated unique induced 4-cycle $\alpha \xleftrightarrow{c_1 Q_1^i c_3} \alpha_1 \xleftrightarrow{c_2 Q_2^j c_4} \beta \xleftrightarrow{c_3 Q_1^i c_1} \alpha_2 \xleftrightarrow{c_4 Q_2^j c_2} \alpha$.

- (2) We are given $\alpha_{G'}(Q_1^i) = c_1$, with edges $\alpha \xleftrightarrow{c_1 Q_1^i c_3} \alpha_1$ and $\alpha \xleftrightarrow{c_1 Q_1^i c_4} \alpha_2$. Thus, $(\alpha_1)_{G'}(Q_1^i) = c_3$ and $(\alpha_2)_{G'}(Q_1^i) = c_4$, while all other parts are colored as in α . Since $c_3 \neq c_4$, α_1 and α_2 differ only on Q_1^i and are thus adjacent via the edge $\alpha_1 \xleftrightarrow{c_3 Q_1^i c_4} \alpha_2$. Together with the edge $\alpha_2 \xleftrightarrow{c_4 Q_1^i c_1} \alpha$ (returning Q_1^i to its color in α), this forms the unique induced 3-cycle $\alpha \xleftrightarrow{c_1 Q_1^i c_3} \alpha_1 \xleftrightarrow{c_3 Q_1^i c_4} \alpha_2 \xleftrightarrow{c_4 Q_1^i c_1} \alpha$.
- (3) We are given $\alpha_{G'}(Q_1^i) = c_1$, $\alpha_{G'}(Q_2^j) = c_2$, with initial edges $\alpha \xleftrightarrow{c_1 Q_1^i c_3} \alpha_1$ and $\alpha \xleftrightarrow{c_2 Q_2^j c_4} \alpha_2$. Here Q_1^i and Q_2^j are disjoint vertex sets, $c_1 \neq c_3$, and $c_2 \neq c_4$. The recolorings $Q_1^i: c_1 \rightarrow c_3$ and $Q_2^j: c_2 \rightarrow c_4$ are independent. If $c_3 \neq c_4$, the reasoning from part (1) applies directly: the independent operations with distinct target colors define the unique induced 4-cycle and its labeling as stated.

Consider the case $c_3 = c_4 = c$. We have $\alpha_{G'} \equiv (Q_1^i: c_1, Q_2^j: c_2)$, $(\alpha_1)_{G'} \equiv (Q_1^i: c, Q_2^j: c_2)$, and $(\alpha_2)_{G'} \equiv (Q_1^i: c_1, Q_2^j: c)$. These vertices complete to a 4-cycle in $\mathcal{L}'_{[\alpha]}$ because the matching 4-cycle exists in $\mathcal{L}_{[\alpha]}$. Let β be the fourth vertex in this 4-cycle $(\alpha, \alpha_1, \beta, \alpha_2, \alpha)$. The α -avoiding path from α_1 to α_2 is $\alpha_1 \rightarrow \beta \rightarrow \alpha_2$. To change from $(\alpha_1)_{G'} \equiv (Q_1^i: c, Q_2^j: c_2)$ to $(\alpha_2)_{G'} \equiv (Q_1^i: c_1, Q_2^j: c)$, part Q_1^i must change from c to c_1 , and part Q_2^j must change from c_2 to c . These two changes must occur over the two edges (α_1, β) and (β, α_2) . There are two ways to assign these changes to the edges:

- (a) Path T_1 : $\alpha_1 \xleftrightarrow{c Q_1^i c_1} \beta \xleftrightarrow{c_2 Q_2^j c} \alpha_2$. Here, β would have Q_1^i colored c_1 and Q_2^j colored c_2 . Thus, $\beta_{G'} \equiv (Q_1^i: c_1, Q_2^j: c_2)$, which is α . This is a contradiction, because β is distinct from α .
 - (b) Path T_2 : $\alpha_1 \xleftrightarrow{c_2 Q_2^j c} \beta \xleftrightarrow{c Q_1^i c_1} \alpha_2$. Here, from $(\alpha_1)_{G'} \equiv (Q_1^i: c, Q_2^j: c_2)$, recoloring $Q_2^j: c_2 \rightarrow c$ yields $\beta_{G'} \equiv (Q_1^i: c, Q_2^j: c)$. This path T_2 avoids α , so this is the correct labeling.
- Therefore, T_2 provides the unique labeling for the α -avoiding path segment $\alpha_1 \rightarrow \beta \rightarrow \alpha_2$. Re-instating c as c_3 (for Q_1^i 's target) and c_4 (for Q_2^j 's target, which is also c_3 in this subcase), the labels become $\alpha_1 \xleftrightarrow{c_2 Q_2^j c_4} \beta$ and $\beta \xleftrightarrow{c_3 Q_1^i c_1} \alpha_2$. This completes the 4-cycle as stated in the lemma: $\alpha \xleftrightarrow{c_1 Q_1^i c_3} \alpha_1 \xleftrightarrow{c_2 Q_2^j c_4} \beta \xleftrightarrow{c_3 Q_1^i c_1} \alpha_2 \xleftrightarrow{c_4 Q_2^j c_2} \alpha$. In retrospect, the properness of β , ensured by the existence of this 4-cycle structure, implies that if $c_3 = c_4$, no edge exists between Q_1^i and Q_2^j . \square

Building upon Lemmas 4.21 and 4.22, we find a natural correspondence between $\mathcal{L}_{[\alpha]}$ and $\mathcal{L}'_{[\alpha]}$. To formalize this correspondence, the following lemma introduces functions describing how a proper k -coloring α_G is lifted to a χ' -coloring $\alpha_{G'}$ on a subgraph of G' isomorphic to G . Recall that H_1, \dots, H_d are the connected components of G , identified and labeled by Algorithm 1 run on $\mathcal{C}_k(G)$ and α . For each H_i , we define two mappings: ψ_i and ϕ_i . The function ψ_i translates colors from the palette $[k]$ used in the G -interpretation of \mathcal{C} to distinct colors in a specific palette $\mathcal{P}_i \subseteq [\chi']$ for the G' -interpretation. The function ϕ_i maps each vertex in H_i to a vertex in G' . We encountered these two functions in Lemma 4.12 at a single abstract link vertex. The following key lemma establishes that ψ_i and ϕ_i can be chosen consistently for every link vertex within the same equivalence class.

Lemma 4.23. Assume Hypothesis 1. For each $i \in [d]$ there exists a subset $\mathcal{P}_i \subseteq [\chi']$ with $|\mathcal{P}_i| = k$ and two functions $\psi_i^{[\alpha]}$ and $\phi_i^{[\alpha]}$ such that:

- (1) $\psi_i^{[\alpha]}: [k] \rightarrow \mathcal{P}_i$ is a bijection.
- (2) $\phi_i^{[\alpha]}: H_i \rightarrow H'_i$ is a graph isomorphism.
- (3) For each labeled edge $\beta \xleftrightarrow{c_1 P_j^i c_2} \gamma$ in $\mathcal{L}_{[\alpha]}$, the corresponding labeled edge in $\mathcal{L}'_{[\alpha]}$ is given by $\beta \xleftrightarrow{\psi_i(c_1)\phi_i(P_j^i)\psi_i(c_2)} \gamma$.

We use $(\psi^{[\alpha]}, \phi^{[\alpha]})$ as a shorthand for the collection of pairs $(\psi_i^{[\alpha]}, \phi_i^{[\alpha]})$ for $1 \leq i \leq d$.

Proof. By Lemma 4.6, there exist vertex-disjoint subgraphs H'_1, \dots, H'_d in G' such that $H'_i \cong H_i$ for each $i \in [d]$. The map $\phi_i^{[\alpha]}: H_i \rightarrow H'_i$ can be constructed by comparing edges incident to α in $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G)$, as in the proof of Lemma 4.6. By Lemma 4.12, for each $i \in [d]$ there exists an injection $\psi_i^{[\alpha]}: [k] \rightarrow [\chi']$ such that $\alpha_{G'}(\phi_i(v)) = \psi_i^{[\alpha]}(\alpha_G(v))$ for each vertex $v \in V(H_i)$. As of yet, $\psi_i^{[\alpha]}$ is not uniquely defined for any of the free colors, those colors that do not appear in α_G . By inspection of incident edges to α in $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G)$, we can define $\psi_i^{[\alpha]}$ to be compatible for free colors. More precisely, for each $i \in [d]$, and each free color c at α_G within H_i , consider β_G obtained from α_G by recoloring all the vertices in a single independent part P_j^i from $\alpha_G(P_j^i)$ to c . We define $\psi_i^{[\alpha]}(c)$ to be $\beta_{G'}(\phi_i(P_j^i))$. This ensures the relation $\beta_{G'}(\phi_i(v)) = \psi_i^{[\alpha]}(\beta_G(v))$ for every β that is a neighbor of α in $\mathcal{L}_{[\alpha]}$. After defining $\mathcal{P}_i = \psi_i^{[\alpha]}([k])$, the map $\psi_i^{[\alpha]}: [k] \rightarrow \mathcal{P}_i$ is a bijection.

By construction, the desired condition (3) holds for all incident edges to α in $\mathcal{L}_{[\alpha]}$, meaning that:

$$\alpha \xleftrightarrow{c_1 P_j^i c_2} \beta \text{ in } \mathcal{L}_{[\alpha]} \quad \implies \quad \alpha \xleftrightarrow{\psi_i(c_1) \phi_i(P_j^i) \psi_i(c_2)} \beta \text{ in } \mathcal{L}'_{[\alpha]}.$$

Choose some neighbor β in $\mathcal{L}'_{[\alpha]}$. Using the same iteration process as Algorithm 3 (Step 5), label all incident edges to β in $\mathcal{L}'_{[\alpha]}$. These edges can be found through a cycle of appropriate length. Then Lemmas 4.21 and 4.22 propagate the desired correspondence (3) from labeled edges incident to α to those incident to β . By continuing this process in a breadth-first ordering as in Algorithm 3 (Step 5), which is correct by Lemma 4.19, all edges in $\mathcal{L}'_{[\alpha]}$ will be labeled using the correspondence given by the pairs $(\psi_i^{[\alpha]}, \phi_i^{[\alpha]})$. \square

Examining the proof of Lemma 4.23, we observe that $\psi^{[\alpha]}$ is constructed precisely in a way to satisfy $\beta_{G'} \circ \phi_i^{[\alpha]} = \psi_i^{[\alpha]} \circ \beta_G$ for each $\beta_G \in [\alpha_G]$. This *compatibility relation* turns out to be important in later arguments, primarily because it lends itself well to computation.

Corollary 4.24. Assume Hypothesis 1. Consider the functions $(\psi_i^{[\alpha]}, \phi_i^{[\alpha]})$ from Lemma 4.23. For each β in $[\alpha]$, the relation $\beta_{G'} \circ \phi_i^{[\alpha]} = \psi_i^{[\alpha]} \circ \beta_G$ holds for each $1 \leq i \leq d$.

The compatibility condition in Corollary 4.24 can be visualized as a commutative diagram.

$$\begin{array}{ccc} V(H_i) & \xrightarrow{\beta_G} & [k] \\ \phi_i^{[\alpha]} \downarrow & & \downarrow \psi_i^{[\alpha]} \\ V(H'_i) & \xrightarrow{\beta_{G'}} & \mathcal{P}_i \end{array}$$

Lemma 4.23 establishes the coloring of the vertices in $V(H'_1) \uplus V(H'_2) \uplus \dots \uplus V(H'_d)$ throughout the equivalence class $[\alpha]$ when viewed inside G' . Let $L \subseteq V(G')$ be the complement of $V(H'_1) \uplus V(H'_2) \uplus \dots \uplus V(H'_d)$. The next corollary asserts that equivalent link colorings agree pointwise on L . We refer to L as the *locked set* associated with the equivalence class $[\alpha]$. For each vertex $\beta \in [\alpha]$, the coloring of L in $\beta_{G'}$ is determined by an orbit representative $\alpha_{G'}$. This fixed assignment of colors to L must be proper with respect to the way H'_1, \dots, H'_d are colored (as derived from β_G), since β exists as a vertex in the coloring graph.

Corollary 4.25. Assume Hypothesis 1. Suppose $\beta \in [\alpha]$. Then $\beta_{G'}(w) = \alpha_{G'}(w)$ for every $w \in L$.

Proof. From Lemma 4.23, α and β share the same functions $(\psi^{[\alpha]}, \phi^{[\alpha]})$, with $\beta_{G'}(\phi_i^{[\alpha]}(v)) = \psi_i^{[\alpha]}(\beta_G(v))$ for every $v \in V(H_i)$ and $i \in [d]$. Since α and β are connected in $\mathcal{L}_{[\alpha]}$, it suffices to reduce (by induction on the length of path between them) to the case when α and β are adjacent in $\mathcal{L}_{[\alpha]}$. In this case, we have an

edge of the form $\alpha \xleftrightarrow{c_1 P_j^i c_2} \beta$ where $P_j^i \subset V(H_i)$ is the unique part on which α_G and β_G differ. Specifically, $\alpha_G(P_j^i) = c_1$, $\beta_G(P_j^i) = c_2$, and $\alpha_G(v) = \beta_G(v)$ for all $v \in V(G) \setminus P_j^i$. Using Corollary 4.24, the colorings $\alpha_{G'}$ and $\beta_{G'}$ satisfy the compatibility relations with the same pair $(\psi^{[\alpha]}, \phi^{[\alpha]})$:

$$\alpha_{G'} \circ \phi_i^{[\alpha]} = \psi_i^{[\alpha]} \circ \alpha_G \quad \text{and} \quad \beta_{G'} \circ \phi_i^{[\alpha]} = \psi_i^{[\alpha]} \circ \beta_G$$

Applying these identities on the input $V(H_i)$, we obtain that $\alpha_{G'}$ and $\beta_{G'}$ differ on a single part $\phi_i^{[\alpha]}(P_j^i)$. By counting the number of edges between the vertices α and β in \mathcal{C} , we see that $\alpha_{G'}$ and $\beta_{G'}$ must agree on all vertices in $V(G') \setminus \phi_i^{[\alpha]}(P_j^i)$. Since L is disjoint from $V(H'_i)$, we get $\alpha_{G'}(w) = \beta_{G'}(w)$ for every $w \in L$. \square

An example of the pair $(\psi^{[\alpha]}, \phi^{[\alpha]})$ and the associated locked set L is illustrated in the top row of Figure 11. We leverage the previous results to deduce more information about the χ' -colorings of G' at the abstract link vertices. From Lemma 4.6, we know that G' contains G as a subgraph where each connected component H_i of G appears as an induced subgraph H'_i . However, it is certainly possible for there to be an edge connecting H'_i and H'_j for $i \neq j$. The following corollary imposes strong restrictions in this case. It concludes that if H'_i and H'_j are connected by an edge, their respective color palettes \mathcal{P}_i and \mathcal{P}_j must be disjoint. Furthermore, no vertex in the locked set L adjacent to a vertex in H'_i can have a color from \mathcal{P}_i . Finally, the locked set L must use every color in the $[\chi']$ palette. Figure 11 depicts these properties. In the statement below, $N(v)$ denotes the set of neighbors of a vertex v in G' .

Corollary 4.26. Assume Hypothesis 1. Let α be an abstract link vertex. Then

- (1) for every $1 \leq i < j \leq d$, if there exist $u \in H'_i, v \in H'_j$ with $uv \in E(G')$, then $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$;
- (2) for every $i \in [d], v \in H'_i$, $\alpha_{G'}(N(v) \cap L) \subseteq [\chi'] \setminus \mathcal{P}_i$;
- (3) $\alpha_{G'}(L) = [\chi']$.

Proof. By Lemma 4.23, all α, β with $[\beta] = [\alpha]$ share the same pair $(\psi^{[\alpha]}, \phi^{[\alpha]})$ defining a fixed partition (H'_1, \dots, H'_d, L) of G' where $H'_i \cong H_i$.

- (1) Let $uv \in E(G')$ with $u \in V(H'_i)$ and $v \in V(H'_j)$ for $i \neq j$. Suppose, to the contrary, that $\mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset$. Take any color $c \in \mathcal{P}_i \cap \mathcal{P}_j$. Within $[\alpha]$ we may independently permute the colors (within k available colors) inside each H_i and H_j . Consequently, we can find a link k -coloring β_G such that the associated χ' -coloring $\beta_{G'}$ simultaneously satisfies: $\beta_{G'}(u) = c$ and $\beta_{G'}(v) = c$. This contradicts the fact that $\beta_{G'}$ is a proper coloring, and $uv \in E(G')$.
- (2) We show that no color $c \in \mathcal{P}_i$ appears on $N(v) \cap L$. Otherwise, choose $w \in N(v) \cap L$ with $\alpha_{G'}(w) = c$. Find $\beta_G \in [\alpha_G]$ such that $\beta_{G'}(v) = c$. This is possible because $v \in H'_i$. Since $\beta_{G'}|_L = \alpha_{G'}|_L$ by Corollary 4.25, we have $\beta_{G'}(w) = \alpha_{G'}(w) = c$. The resulting equality $\beta_{G'}(v) = \beta_{G'}(w)$ leads to a contradiction as $\beta_{G'}$ is a proper coloring and $vw \in E(G')$.
- (3) Suppose $\tilde{c} \in [\chi']$ is not present in $\alpha_{G'}(L)$. Find $\beta \in [\alpha]$ such that $\tilde{c} \notin \beta_{G'}(V(H'_i))$ for each $i \in [d]$. This is possible, as $\beta_{G'}(V(H'_i)) = \psi_i^{[\alpha]}(\beta_G(V(H_i)))$ and G has surplus colors. Now, $\tilde{c} \notin \alpha_{G'}(L) = \beta_{G'}(L)$. As $\tilde{c} \notin \beta_{G'}(V(G'))$, it follows that $\beta_{G'}$ is a proper $(\chi' - 1)$ -coloring of G' , a contradiction. \square

As a final result in this subsection, we use the accumulated information regarding the proper coloring $\alpha_{G'}$ to apply the pair $(\psi^{[\alpha]}, \phi^{[\alpha]})$ to any abstract link vertex, not just those in $[\alpha]$. In particular, we can convert *any* link k -coloring of G to a proper χ' -coloring of G' by applying the pair $(\psi^{[\alpha]}, \phi^{[\alpha]})$, while retaining the colors of the locked set L as in $\alpha_{G'}$. Figure 11 shows an example of applying $(\psi^{[\alpha]}, \phi^{[\alpha]})$ to an abstract link vertex not in $[\alpha]$ and still obtaining a proper coloring that is itself an abstract link vertex. This “surgery” technique produces *many* abstract link vertices from an equivalence class $[\alpha]$ of a *single* abstract link vertex. This process will demonstrate that the set of abstract link vertices of $\mathcal{C}_k(G)$ cannot be in bijection with the set of abstract link vertices of $\mathcal{C}_{\chi'}(G')$, ultimately showing that Hypothesis 1 is untenable.

Lemma 4.27. Assume Hypothesis 1. Let α, β be two abstract link vertices in \mathcal{C} such that $\beta \notin [\alpha]$. Let $(\psi^{[\alpha]}, \phi^{[\alpha]})$ be a pair for $[\alpha]$ from Lemma 4.23. Define $\tilde{\beta}_{G'}: V(G') \rightarrow [\chi']$ by specifying $\tilde{\beta}_{G'}(\phi^{[\alpha]}(v)) = \psi^{[\alpha]}(\beta_G(v))$ for $v \in V(G)$ and $\tilde{\beta}_{G'}(w) = \alpha_{G'}(w)$ for $w \in L$. Then $\tilde{\beta}_{G'}$ is a proper coloring of G' . Moreover, the corresponding vertex $\tilde{\beta} \in \mathcal{C}$ (which may or may not be β) is an abstract link vertex satisfying $\tilde{\beta} \notin [\alpha]$.

Proof. We first establish that $\tilde{\beta}_{G'}$ is a proper χ' -coloring. The domain of $\tilde{\beta}_{G'}$ is

$$V(G') = V(H'_1) \uplus \dots \uplus V(H'_d) \uplus L.$$

We carefully check properness of $\tilde{\beta}_{G'}$ on the set $E(G')$ by separating the edges into four types.

- *Edges within H'_i :* For each $u \in H'_i$, we have $\tilde{\beta}_{G'}(u) = \beta_G(h_i^{-1}(u))$. Since β_G is a proper coloring and $h_i: H_i \rightarrow U_i$ is an isomorphism, $\tilde{\beta}_{G'}$ preserves properness on $E(H'_i)$.
- *Edges between H'_i and H'_j for $i \neq j$:* Suppose $uv \in E(G')$ with $u \in H'_i$ and $v \in H'_j$. Corollary 4.26(1) implies $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$. Since $\tilde{\beta}_{G'}(u) \in \mathcal{P}_i$ and $\tilde{\beta}_{G'}(v) \in \mathcal{P}_j$, we deduce that $\tilde{\beta}_{G'}(u) \neq \tilde{\beta}_{G'}(v)$.

- *Edges between H'_i and L :* Suppose $uw \in E(G')$ with $u \in H'_i$ and $w \in L$. By Corollary 4.26(2), the image set $\alpha_{G'}(N(u) \cap L) = \tilde{\beta}_{G'}(N(u) \cap L)$ is disjoint from \mathcal{P}_i . Since $w \in N(u) \cap L$ and $\tilde{\beta}_{G'}(u) \in \mathcal{P}_i$, it follows that $\tilde{\beta}_{G'}(w) \neq \tilde{\beta}_{G'}(u)$.
- *Edges within L :* The coloring $\tilde{\beta}_{G'}|_L$ is identical to $\alpha_{G'}|_L$. Since $\alpha_{G'}$ is a proper coloring, $\tilde{\beta}_{G'}$ is also proper on the subgraph induced by L .

Next, we argue that $\tilde{\beta} \in \mathcal{C}$ (the vertex corresponding to the χ' -coloring $\tilde{\beta}_{G'}$) is an abstract link vertex. When verifying that β is an abstract link vertex, Algorithm 1 checks the clique-neighborhood around the vertex β_G . The same check applies to $\tilde{\beta}$ because $\phi^{[\alpha]}$ is a graph isomorphism, $\psi^{[\alpha]}$ is a bijection, and $\tilde{\beta}_{G'} \circ \phi^{[\alpha]}$ and $\psi^{[\alpha]} \circ \beta_G$ agree on $V(G)$. Therefore, $\tilde{\beta}$ is also an abstract link vertex. It remains to show that $\tilde{\beta} \notin [\alpha]$. The relation $\tilde{\beta}_{G'}(\phi^{[\alpha]}(v)) = \psi^{[\alpha]}(\beta_G(v))$ for all $v \in V(G)$ implies that the corresponding k -coloring $\tilde{\beta}_G$ coincides with β_G . Since $\beta \notin [\alpha]$, it follows that $\tilde{\beta} \notin [\alpha]$, as desired. \square

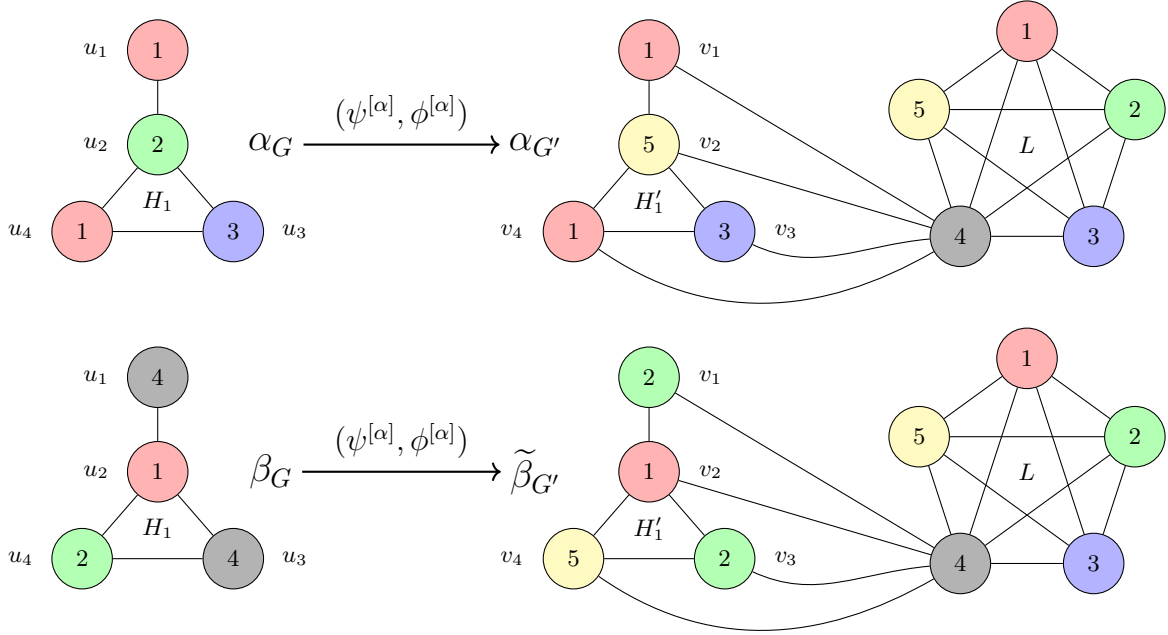


FIGURE 11. Proper colorings of two graphs G , with $k = 4 > \chi$, and G' , with $k' = 5 = \chi'$. Consider the pair of functions $(\psi^{[\alpha]}, \phi^{[\alpha]})$ defined by $\phi(u_i) = v_i$ and $\psi(1) = 1, \psi(2) = 5, \psi(3) = 3$, and $\psi(4) = 2$ with the palette $\mathcal{P} = \{1, 2, 3, 5\}$. The top row shows a proper coloring $\alpha_{G'}$ of G' that relates to α_G via the map $(\psi^{[\alpha]}, \phi^{[\alpha]})$. These two vertices look identical locally in their coloring graphs. The bottom row shows that this pair $(\psi^{[\alpha]}, \phi^{[\alpha]})$, when applied to an abstract link vertex $\beta \notin [\alpha]$, still creates a proper coloring $\tilde{\beta}_{G'}$ that locally resembles an abstract link vertex. We note that G and G' do not have equivalent coloring graphs; this example merely illustrates Properties 2 and 3 in Corollary 4.26 and Lemma 4.27.

4.4. Abstract link vertices of $\mathcal{C}_k(G), k > \chi(G)$ and $\mathcal{C}_{\chi'}(G'), \chi' = \chi(G')$ can never be in bijection if $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$. In this final subsection, we use the structural results accumulated by Algorithms 1, 2, and 3 to establish our main result: if a coloring graph \mathcal{C} arises from a graph with surplus colors, then \mathcal{C} does *not* arise from any other base graph.

Theorem 4.4 establishes that if $\mathcal{C}_k(G)$ with $k > \chi(G)$ is isomorphic to $\mathcal{C}_{k'}(G')$, then $k' = \chi' := \chi(G')$ (meaning G' has no surplus colors). Theorem 4.28 below proves this remaining scenario is impossible. Assuming, for contradiction, that $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$, we use properties of abstract link vertices. Our analysis

reveals that if the shared graph \mathcal{C} is interpreted as $\mathcal{C}_{\chi'}(G')$, it must contain $\chi'!$ times the number of abstract link vertices defined by its structure as $\mathcal{C}_k(G)$. This discrepancy yields a contradiction, ruling out the isomorphism. The theorem upgrades Corollary 4.9 by removing the dependence on the number of vertices.

Theorem 4.28. *Given a graph G and $k > \chi(G)$, there is no graph G' such that $\mathcal{C}_k(G) \cong \mathcal{C}_{\chi(G')}(G')$.*

Proof. Assume, to the contrary, that $\mathcal{C} \cong \mathcal{C}_k(G) \cong \mathcal{C}_{\chi'}(G')$ for some graphs G and G' , where $k > \chi(G)$ and $\chi' = \chi(G')$. First, we show that $\chi' > 1$. If $\chi' = 1$, then G' is the edgeless graph N_r (with $r \geq 1$ vertices). The 1-coloring graph $\mathcal{C}_1(N_r)$ contains exactly one vertex (representing the single coloring that assigns color 1 to all vertices). In contrast, the number of proper k -colorings of G is at least $k! > \chi(G)! \geq 1$. Thus, $\mathcal{C}_k(G)$ has at least two vertices, contradicting that $\mathcal{C}_1(N_r)$ has only one vertex. So, we must have $\chi' > 1$.

Let A be the set of abstract link vertices in \mathcal{C} . Since $\mathcal{C} \cong \mathcal{C}_k(G)$ where $k > \chi(G)$, abstract link vertices exist by Lemma 4.3, so $|A| \geq 1$. Fix a vertex $\alpha \in A$, and let $(\psi^{[\alpha]}, \phi^{[\alpha]})$ be a pair associated with the equivalence class $[\alpha]$ as in Lemma 4.23. As before, let $L \subseteq V(G')$ denote the locked set for $(\psi^{[\alpha]}, \phi^{[\alpha]})$. For every $\beta \in A$, define $\tilde{\beta}_{G'}$ by $\tilde{\beta}_{G'}(\phi^{[\alpha]}(v)) = \psi^{[\alpha]}(\beta_G(v))$ for $v \in V(G)$ and $\tilde{\beta}_{G'}(w) = \alpha_{G'}(w)$ for $w \in L$. If $\beta \in [\alpha]$, we have $\tilde{\beta}_{G'} = \beta_{G'}$ by Corollaries 4.24 and 4.25, which corresponds to an abstract link vertex. If $\beta \notin [\alpha]$, then $\tilde{\beta}_{G'}$ is a proper χ' -coloring of G' by Lemma 4.27, corresponding to an abstract link vertex in \mathcal{C} . The construction produces $|A|$ distinct proper χ' -colorings of G' , each representing a different abstract link vertex in \mathcal{C} , that all agree pointwise on the coloring of L .

For each such coloring $\tilde{\beta}_{G'}$, consider the set of $\chi'!$ colorings obtained by applying every permutation $\sigma \in S_{\chi'}$ to the global color set $[\chi']$. Each resulting coloring $\sigma \circ \tilde{\beta}_{G'}$ is a proper χ' -coloring of G' . Since $\tilde{\beta}_{G'}$ corresponds to an abstract link vertex (a property preserved under permutation of colors), each $\sigma \circ \tilde{\beta}_{G'}$ must also correspond to an abstract link vertex. These $\chi'!$ colorings generated from a single $\tilde{\beta}_{G'}$ are all pairwise distinct. Indeed, if $\sigma \circ \tilde{\beta}_{G'} = \tau \circ \tilde{\beta}_{G'}$, then their restrictions to L are equal: $\sigma(\alpha_{G'}|_L) = \tau(\alpha_{G'}|_L)$. Since $\alpha_{G'}(L) = [\chi']$ by Corollary 4.26 (3), we deduce that $\sigma = \tau$.

Moreover, the resulting sets of $\chi'!$ colorings generated by two *distinct* initial colorings, say $\tilde{\beta}_{G'}$ and $\tilde{\gamma}_{G'}$ (where $\beta, \gamma \in A$ with $\beta \neq \gamma$), are themselves disjoint. Indeed, if $\sigma \circ \tilde{\beta}_{G'} = \tau \circ \tilde{\gamma}_{G'}$, then considering the colors on L , we again get $\sigma(\alpha_{G'}|_L) = \tau(\gamma_{G'}|_L)$. The same reasoning from the previous paragraph yields $\sigma = \tau$. This implies $\tilde{\beta}_{G'} = \tilde{\gamma}_{G'}$, and consequently, $\tilde{\beta}_{G'}(\phi^{[\alpha]}(v)) = \tilde{\gamma}_{G'}(\phi^{[\alpha]}(v))$ for all $v \in V(G)$. By definition, $\psi^{[\alpha]}(\beta_G(v)) = \psi^{[\alpha]}(\gamma_G(v))$ for all $v \in V(G)$. As $\psi^{[\alpha]}$ is injective, $\beta_G(v) = \gamma_G(v)$ for all $v \in V(G)$, leading to $\beta_G = \gamma_G$. This equality between k -colorings of G implies $\beta = \gamma$ as vertices in \mathcal{C} , yielding a contradiction.

Therefore, the overall process generates exactly $|A| \cdot \chi'!$ distinct χ' -colorings of G , each corresponding to an abstract link vertex in \mathcal{C} . Hence, the number of abstract link vertices in $\mathcal{C} \cong \mathcal{C}_{\chi'}(G')$ is at least $|A| \cdot \chi'!$. By definition, the number of abstract link vertices in $\mathcal{C} \cong \mathcal{C}_k(G)$ is exactly $|A|$. Comparing the two counts, we obtain $|A| \geq |A| \cdot \chi'!$. This inequality yields a final contradiction, because $|A| \geq 1$ and $\chi' > 1$. \square

The progression of results in Section 4, which collectively establish Theorem 4.28, can be visualized in the flowchart presented in Figure 12. Combining Theorem 4.4 and Theorem 4.28, our main theorem Theorem 1.1 is now proved. We conclude that coloring graphs with surplus colors are complete graph invariants.

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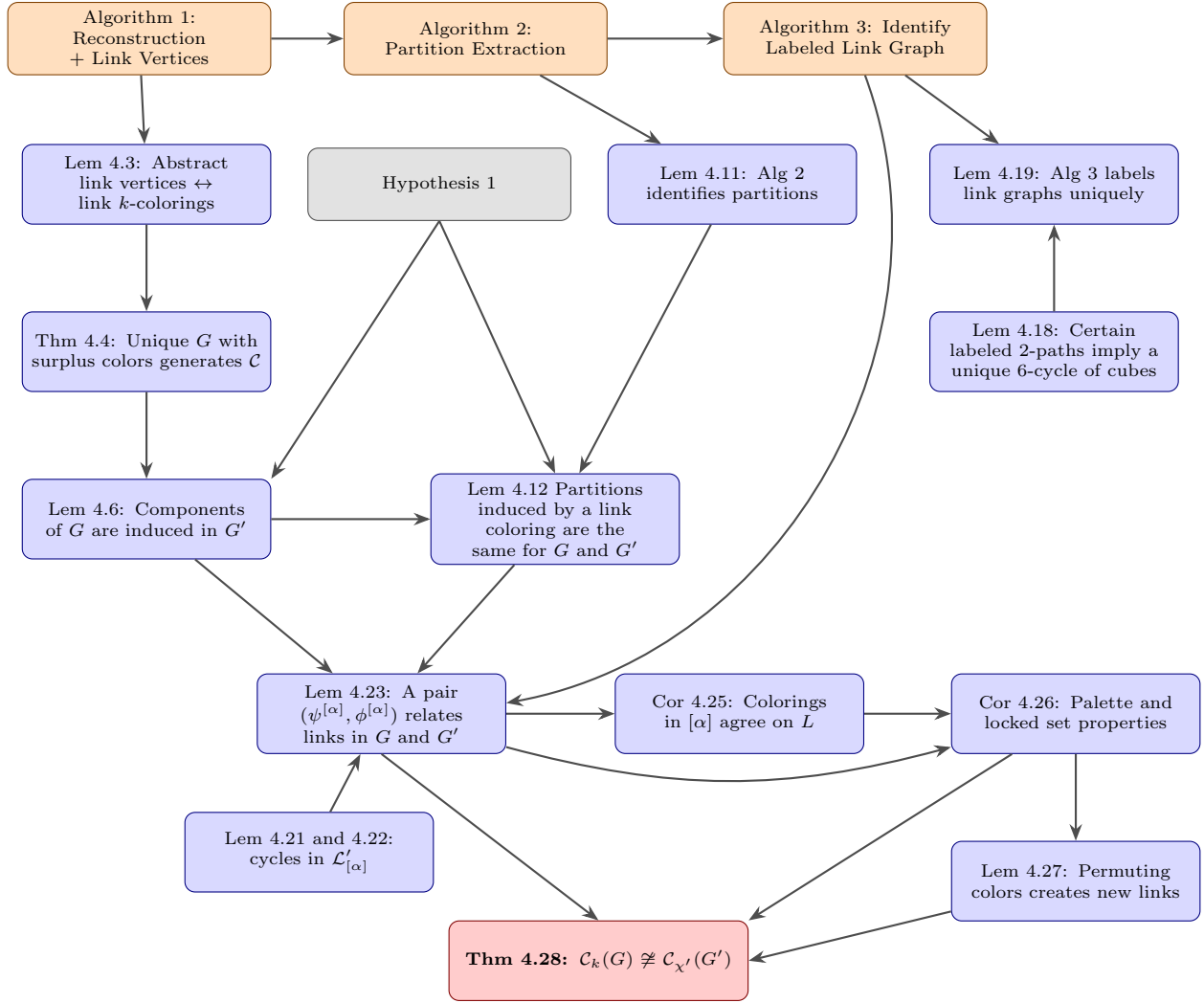


FIGURE 12. Flowchart for Section 4

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