COLLISION OF ORBITS FOR FAMILIES OF POLYNOMIALS DEFINED OVER FIELDS OF POSITIVE CHARACTERISTIC

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ABSTRACT. Let L be a field of positive characteristic p with a fixed algebraic closure $\overline{L}$, and let $\alpha_1, \alpha_2, \beta \in L$. For an integer $d \geq 2$, we consider the family of polynomials $f_\lambda(z) := z^d + \lambda$, parameterized by $\lambda \in \overline{L}$. Define $C(\alpha_1, \alpha_2; \beta)$ to be the set of all $\lambda \in \overline{L}$ for which there exist $m, n \in \mathbb{N}$ such that $f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta$. In other words, $C(\alpha_1, \alpha_2; \beta)$ consists of all $\lambda \in \overline{L}$ with the property that the orbit of α_1 collides with the orbit of α_2 under the same polynomial f_λ precisely at the point β. Assuming $\alpha_1, \alpha_2, \beta$ are not all contained in a finite subfield of L, we provide explicit necessary and sufficient conditions under which $C(\alpha_1, \alpha_2; \beta)$ is infinite. We also discuss the remaining case where $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$ and provide ample computational data that suggest a somewhat surprising conjecture. Our problem fits into a long series of questions in the area of unlikely intersections in arithmetic dynamics, which have been primarily studied over fields of characteristic 0. Working in characteristic p adds significant difficulties, but also reveals the subtlety of our problem, especially when some of the points lie in a finite field or when d is a power of p.

1. INTRODUCTION

1.1. **Notation.** Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers. For each field K, we denote by $\overline{K}$ an algebraic closure of K; if K_0 is the prime subfield of K, we let $\overline{K_0}$ be its algebraic closure inside $\overline{K}$. We mention standard definitions from algebraic dynamics. For any self-map Φ on a quasiprojective variety X and for any $n \in \mathbb{N}$, we let Φ^n be the n-th compositional iterate of Φ; by convention, Φ^0 is the identity map. We define the *(strict) forward orbit* of a point $\alpha \in X$ as the set of all points $\Phi^n(\alpha)$, for $n \geq 1$. Similarly, we define the *(strict) backward orbit* of α as the set $\overline{\mathcal{O}}_\Phi(\alpha)$, which consists of all $\gamma \in X$ such that there exists $m \in \mathbb{N}$ with $\Phi^m(\gamma) = \alpha$. A point α is *preperiodic* under the action of Φ if there exist $0 \leq m < n$ such that $\Phi^m(\alpha) = \Phi^n(\alpha)$; if $m = 0$, then α is *periodic*.

1.2. **The unlikely intersection principle in arithmetic dynamics.** Several central questions in arithmetic geometry are rooted in the principle of unlikely intersections; for more details, we refer the reader to the excellent book of Zannier [Zan12]. Over the past 30 years, there has been considerable research in algebraic dynamics on questions framed by this general principle; we provide a couple of prominent examples below.

The dynamical Mordell-Lang (DML) conjecture considers a quasiprojective variety X with an endomorphism Φ (over a field L of characteristic 0), an irreducible curve $C \subset X$, and a point $\alpha \in X(L)$. The conjecture predicts that the *unlikely* occurrence of infinitely many points in $\mathcal{O}_\Phi(\alpha) \cap C$ *must* be explained only by the fact that C is periodic under the action of Φ (i.e., $\Phi^n(C) \subseteq C$ for some $n \in \mathbb{N}$). For more details on the DML conjecture and a survey of some of the partial results, we refer the reader to [BGT16].

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In a different direction, we describe the problem of *simultaneously preperiodic points* (which was itself motivated by [MZ10, MZ12]) for an algebraic family of polynomials. Consider a family of polynomials f_λ of degree $d \geq 2$, whose coefficients depend polynomially on $\lambda \in \mathbb{C}$, and two points $\alpha, \beta \in \mathbb{C}$. The *unlikely* existence of infinitely many parameters $\lambda \in \mathbb{C}$ such that both α and β are preperiodic for f_λ can only occur if α and β are *dynamically related* with respect to the *entire* family of polynomials f_λ (see [BD13, Theorem 1.3], which extends the previous results of [BD11, GHT13]). For a broader survey of recent work and new research directions on unlikely intersection questions in arithmetic dynamics, we refer the reader to [BIJMST19].

The vast majority of the proven results and open problems in arithmetic dynamics concern the algebraic dynamical systems defined over fields of characteristic 0 (see [BIJMST19] and the references therein). It is only recently that several outstanding conjectures have been considered in positive characteristic; in each case, new features emerge for the dynamical systems in characteristic p. These new intricacies are due to the presence of the Frobenius map (similar to the isotriviality issues appearing in [CHT23]) and the existence of additive polynomials of degree greater than one (which lead to new unlikely intersection questions as in [BM17, BM22]). Indeed, the DML conjecture (see [BGT15, CGSZ21, Ghi19, GOSS21, XY25, XY, Yan24]), the problem of simultaneous preperiodic points for an algebraic family of polynomials (see [Ghi]), and the Zariski Dense Orbit conjecture (see [MS14] for its formulation over fields of characteristic 0 and [GS23a, GS23b] for the problem over fields of characteristic p) are all more subtle when studied in a positive characteristic setting.

Motivated by [BD11], the second author studied the following problem in [Ghi]. Given an integer $d \geq 2$, a field L of characteristic p, and points $\alpha_1, \alpha_2 \in L$, we obtained (see [Ghi, Theorem 1.1]) necessary and sufficient conditions for the existence of infinitely many $\lambda \in \overline{L}$ such that both α_1 and α_2 are preperiodic for the polynomial $f_\lambda(z) := z^d + \lambda$. In this article, we extend the problem studied in [Ghi] to the following setting. Working again with the family of polynomials $f_\lambda(z) := z^d + \lambda$ and given *two starting points* α_1, α_2, we study the conditions under which the strict forward orbits of these starting points contain a given *target point* β. We refer to this setting as the *colliding orbits problem*. A similar question of colliding orbits was previously studied in the context of Drinfeld modules (see [Ghi24]).

1.3. **Our main result.** In the present paper, we prove the following statement.

Theorem 1.1. *Let L be a field of characteristic $p > 0$, let $\alpha_1, \alpha_2, \beta \in L$ and let $d \geq 2$ be an integer. Consider the family of polynomials $f_\lambda(z) = z^d + \lambda$, parameterized by $\lambda \in \overline{L}$. Assume $\alpha_1, \alpha_2, \beta$ are not all contained in a finite subfield of L. Then the set*

$$C(\alpha_1, \alpha_2; \beta) := \left\{\lambda \in \overline{L} : \text{ there exist } m, n \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta\right\} \tag{1.3.1}$$

is infinite if and only if exactly one of the following two conditions holds:

- *(A) $\alpha_1^d = \alpha_2^d$;*
- *(B) $d = p^\ell$ for some $\ell \in \mathbb{N}$ and there exists a finite subfield $\mathbb{F}_q \subset L$ such that $\delta_1 := \alpha_2 - \alpha_1 \in \mathbb{F}_q^*$ and $\delta_2 := \beta - \alpha_1 \in \mathbb{F}_q$. Furthermore, the system of two equations:*

$$\begin{cases} \delta_1 &= \sum_{i=0}^{s_1-1} \gamma^{p^{ik\ell}} \\ \delta_2 &= \sum_{i=0}^{s_2-1} \gamma^{p^{ik\ell}} \end{cases} \tag{1.3.2}$$

has a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Remark 1.2. At first glance, alternative (B) in Theorem 1.1 appears asymmetrical, as the roles of α_1 and α_2 apparently cannot be interchanged. However, as we will show in Lemma 8.1, we can always reduce the problem to the system (1.3.2), where the constants are defined relative to α_1 as $\delta_1 := \alpha_2 - \alpha_1$ and $\delta_2 := \beta - \alpha_1$.

Furthermore, the condition $\alpha_1 \neq \alpha_2$ in alternative (B) (since $\delta_1 \in \mathbb{F}_q^*$) ensures that alternatives (A) and (B) are mutually exclusive. For $d = p^\ell$, alternative (A) becomes $\alpha_1^{p^\ell} = \alpha_2^{p^\ell}$, which is equivalent to $\alpha_1 = \alpha_2$ in characteristic p.

By definition, for each $\lambda \in C(\alpha_1, \alpha_2; \beta)$, we have $\beta \in \mathcal{O}_{f_\lambda}(\alpha_1) \cap \mathcal{O}_{f_\lambda}(\alpha_2)$, i.e., the orbits of α_1 and α_2 under the action of f_λ collide at the point β. The existence of infinitely many such parameters λ is clearly an unlikely event, which we show occurs only when α_1, α_2, and β are dynamically related. Indeed, condition (A) in Theorem 1.1 states that the orbits of α_1 and α_2 *merge after one iteration for all parameters* λ, i.e., $\mathcal{O}_{f_\lambda}(\alpha_1) = \mathcal{O}_{f_\lambda}(\alpha_2)$.

Condition (B) provides a more subtle dynamical relation between our three points when $d = p^\ell$. For any $\lambda \in \overline{L}$, a suitable iterate of the polynomial $f_\lambda(z) = z^{p^\ell} + \lambda$ commutes with any given translation polynomial $T_\xi(z) := z + \xi$ for $\xi \in \overline{\mathbb{F}}_p$. Specifically, if $\xi \in \mathbb{F}_{p^{r\ell}}$ then $f_\lambda^r \circ T_\xi = T_\xi \circ f_\lambda^r$ (see equation (8.3.1) for f_λ^r). Since $\alpha_1, \alpha_2, \beta$ differ by elements from $\overline{\mathbb{F}}_p$ in condition (B), this commutativity establishes that three points are indeed dynamically related with respect to our entire family of polynomials $f_\lambda(z)$. The precise role of the system (1.3.2) is explained in Section 8 (see Proposition 8.3).

For the converse direction of Theorem 1.1, we prove slightly stronger statements that also cover the case where $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$ (see Theorems 2.2 and 2.4). In Section 9, based on extensive numerical experiments, we propose Conjecture 9.1 to address the case where $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$ and d is not a power of p. In Section 2, we also formulate a general Conjecture 2.6 regarding colliding orbits for families of polynomials.

1.4. **Further connections for our results.** Another motivation for our Theorem 1.1 arises from [GHT18]. A special case of [GHT18, Theorem 1.1] can be formulated for the Legendre family of elliptic curves E_t given by the equation $y^2 = x(x-1)(x-t)$ as we vary $t \in \overline{\mathbb{Q}}$. We denote by $[k]_t$ the multiplication-by-k map (for any $k \in \mathbb{Z}$) on E_t. We also let $\mathcal{E}$ be the elliptic surface corresponding to the Legendre family and let $[k]$ be the corresponding multiplication-by-k map on $\mathcal{E}$. Any section $\mathcal{P}$ on $\mathcal{E}$ corresponds to an algebraic family of points $P_t \in E_t$ (for all but finitely many $t \in \overline{\mathbb{Q}}$). Then [GHT18, Theorem 1.1] asserts that for any 3 sections $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ on $\mathcal{E}$, if the set

$$C(\mathcal{P}, \mathcal{Q}; \mathcal{R}) := \left\{ t \in \overline{\mathbb{Q}} : \text{ there exist } m, n \in \mathbb{Z} \text{ such that } [m]_t(P_t) = [n]_t(Q_t) = R_t \right\}$$

is infinite, then at least one of the following two conditions must hold:

- there exist $a, b \in \mathbb{Z}$, not both equal to 0, such that $[a](\mathcal{P}) = [b](\mathcal{Q})$;
- there exists $c \in \mathbb{Z}$ such that either $[c](\mathcal{P}) = \mathcal{R}$ or $[c](\mathcal{Q}) = \mathcal{R}$.

In other words, if there exist infinitely many $t \in \overline{\mathbb{Q}}$ such that the cyclic groups generated by both P_t and Q_t contain R_t (in E_t), then at least 2 of the 3 sections must be linearly dependent globally, on the elliptic surface. Once again, we encounter the principle of unlikely intersections: the existence of infinitely many $t \in \overline{\mathbb{Q}}$ for which R_t is contained in the orbits of *both* P_t and Q_t under the action of $\mathbb{Z}$ is explained *only* by a global dynamical relation between their corresponding sections, $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$.

Remark 1.3. The motivation for [GHT18, Theorem 1.1] itself comes from the work of Hsia and Tucker [HT17] on a dynamical analogue of the classical *GCD*-problem. Given multiplicatively independent $a, b \in \mathbb{N}$, the classical *GCD*-problem (solved by Bugeaud, Corvaja, and Zannier [BCZ03]) provides good bounds for $\gcd(a^n - 1, b^n - 1)$ as a function of $n \in \mathbb{N}$. Several extensions of the result from [BCZ03] were obtained, going beyond the classical setting and studying the question for function fields, including in positive characteristic (see [AR04, CZ11, CZ13, GHT17]). Our own work in Theorem 1.1 can be viewed in a similar light, as it is essentially a question about the greatest common divisor of orbits generated by two algebraic families of polynomials (see Remark 3.1 and Section 9).

The hypothesis in Theorem 1.1 that β is contained in the forward orbits of both α_1 and α_2 (under the action of f_λ) can be restated as follows: α_1 and α_2 are contained in the backward orbit of β (under the action of f_λ). The study of backward orbits and their associated arboreal Galois groups is a topic of great interest (see [BFHJY17, BJ19, BJ07] for only a sample of the work in this area). It is also worth noting that the case of a polynomial whose critical points have colliding orbits represents a special case in the study of the corresponding arboreal Galois groups (see [BD24, BDNSWW25]).

1.5. **Plan for our paper.** In Section 2, we present the general strategy for our proof of Theorem 1.1. We split the content of Theorem 1.1 into three distinct results: Theorems 2.1, 2.2, and 2.4. In Section 2, we also outline future research directions in the area of colliding orbits by formulating Conjecture 2.6 for arbitrary families of polynomials. For a brief discussion of the characteristic 0 case, see Remark 2.8.

In Sections 3, 4 and 5, we establish useful preliminary results, which are then employed in the proof of Theorem 2.1. We finish its proof in Section 6, thus completing the direct implication in Theorem 1.1.

In Section 7, we complete the proof of Theorem 2.4, while in Section 8, we prove Theorem 2.2. Combined, these two results provide the converse implication in Theorem 1.1.

We conclude by discussing in Section 9 the case of colliding orbits when the starting points α_1, α_2 and the target point β all live in a finite field. We believe (see Conjecture 9.1) that in this case, the set $C(\alpha_1, \alpha_2; \beta)$ is infinite, provided d is not a power of p. We have ample numerical evidence to support this conjecture; furthermore, we formulate additional questions and conjectures all predicting a higher-than-expected frequency of unlikely intersections when the entire dynamical system is defined over $\overline{\mathbb{F}}_p$.

2. Strategy for our proof of Theorem 1.1 and further extensions

In Subsection 2.1, we state Theorem 2.1 and then explain its proof strategy. In Subsection 2.2, we state Theorems 2.2 and 2.4 and briefly mention key ideas in their proofs. We discuss possible extensions of our results in Subsection 2.3.

2.1. **The direct implication in Theorem 1.1.** We will prove the following.

Theorem 2.1. *Let L be a field of prime characteristic p, let $\alpha_1, \alpha_2, \beta \in L$ and let $d \geq 2$ be an integer. Consider the family of polynomials $f_\lambda(z) = z^d + \lambda$, parameterized by $\lambda \in \overline{L}$. Let $\overline{\mathbb{F}}_p$ denote the algebraic closure of $\mathbb{F}_p$ inside $\overline{L}$. If the set*

$$C(\alpha_1, \alpha_2; \beta) := \left\{\lambda \in \overline{L} : \text{ there exist } m, n \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta\right\} \tag{2.1.1}$$

is infinite, then at least one of the following conditions must hold:

(i) $\alpha_1^d = \alpha_2^d$.
(ii) $d = p^\ell$ *for some* $\ell \in \mathbb{N}$, *and* $\alpha_1 - \beta, \alpha_2 - \beta \in \overline{\mathbb{F}}_p$.
(iii) $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$.

We now outline the strategy for proving Theorem 2.1. First, assuming condition (iii) does *not* hold, the infinitude of the set $C(\alpha_1, \alpha_2; \beta)$ implies that either condition (i) is satisfied, or $\operatorname{trdeg}_{\mathbb{F}_p}(\mathbb{F}_p(\alpha_1, \alpha_2, \beta)) = 1$; this reduction is proved in Subsection 3.2 through a series of Lemmas and Propositions. This allows us to set up the height machine in Section 4; in particular, we obtain a key technical statement (see Proposition 5.2) regarding the variation of the (global) canonical height $\widehat{h}_\lambda(\alpha)$ of a point α (associated to a polynomial f_λ from our family) compared to the Weil heights of λ and α. In turn, Proposition 5.2 allows us to show (see Proposition 6.2) that there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq C(\alpha_1, \alpha_2; \beta)$ such that

$$\lim_{k\to\infty} \widehat{h}_{\lambda_k}(\alpha_1) = \lim_{k\to\infty} \widehat{h}_{\lambda_k}(\alpha_2) = 0. \tag{2.1.2}$$

Equation (2.1.2) is the crucial hypothesis needed to apply Theorem 6.3, which leads to an equality of the canonical heights of α_1 and α_2 with respect to *each* polynomial f_λ. Then Theorem 6.9 immediately provides the desired conclusion in Theorem 2.1 when d is not a power of p. The remaining case, where $d = p^\ell$ for some $\ell \in \mathbb{N}$, requires a more in-depth analysis to arrive at condition (ii) in Theorem 2.1 (see Propositions 6.4 and 6.8). To obtain the more precise information from condition (B) in Theorem 1.1 regarding system (1.3.2), we will rely on Theorem 2.2.

2.2. **The converse implication in Theorem 1.1.** In Section 8, we prove the following result, which provides (along with Theorem 2.1) the full conclusion in Theorem 1.1 when $d = p^\ell$.

Theorem 2.2. *Let L be a field of characteristic p, let $\alpha_1, \alpha_2, \beta \in L$ with $\alpha_1 \neq \alpha_2$, and let $d = p^\ell$ for some $\ell \in \mathbb{N}$. For each $\lambda \in \overline{L}$, we let $f_\lambda(z) = z^d + \lambda$. Consider the set*

$$C(\alpha_1, \alpha_2; \beta) = \left\{\lambda \in \overline{L} : \text{ there exist } m, n \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta\right\}$$

and let $\delta_1 := \alpha_2 - \alpha_1$ and $\delta_2 := \beta - \alpha_1$. Assume there exists a finite subfield $\mathbb{F}_q \subseteq L$ such that

$$\delta_1 \in \mathbb{F}_q^* \text{ and } \delta_2 \in \mathbb{F}_q. \tag{2.2.1}$$

Then the set $C(\alpha_1, \alpha_2; \beta)$ is infinite if the system of two equations:

$$\begin{cases} \delta_1 &= \sum_{i=0}^{s_1-1} \gamma^{p^{ik\ell}} \\ \delta_2 &= \sum_{i=0}^{s_2-1} \gamma^{p^{ik\ell}} \end{cases} \tag{2.2.2}$$

has a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Moreover, the set $C(\alpha_1, \alpha_2; \beta)$ is empty if the system (2.2.2) has no solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Remark 2.3. It is interesting that under the hypotheses (2.2.1) of Theorem 2.2, either $C(\alpha_1, \alpha_2; \beta)$ is infinite or empty. Also, in this case, for *each* $\lambda \in C(\alpha_1, \alpha_2; \beta)$, we have that $\alpha_1, \alpha_2, \beta$ are *all* preperiodic under the action of f_λ (see Lemmas 6.6 and 6.7).

On the other hand, under the assumption that $d = p^\ell$, it could be that $C(\alpha_1, \alpha_2; \beta)$ is finite and nonempty, but this can only occur if either $\alpha_1 - \alpha_2$ or $\alpha_1 - \beta$ is *transcendental* over $\mathbb{F}_p$.

The following result shows that condition (A) in Theorem 1.1 implies the existence of infinitely many parameters $\lambda \in C(\alpha_1, \alpha_2; \beta)$ (see (1.3.1)).

Theorem 2.4. *Let L be a field of characteristic p, let $\alpha, \beta \in L$, let $d \geq 2$ be an integer, and let $f_\lambda(z) = z^d + \lambda$ be a family of polynomials parameterized by $\lambda \in \overline{L}$. Then there exist infinitely many $\lambda \in \overline{L}$ such that $f_\lambda^m(\alpha) = \beta$ for some $m \in \mathbb{N}$ (where m depends on λ).*

Indeed, under the condition (A) from Theorem 1.1 that $\alpha_1^d = \alpha_2^d$, we have that $f_\lambda^n(\alpha_1) = f_\lambda^n(\alpha_2)$ for *all* $n \in \mathbb{N}$. Applying Theorem 2.4 to starting point α_1 and target point β, we obtain that the set $C(\alpha_1, \alpha_2; \beta)$ from equation (1.3.1) must be infinite.

In Section 7, we prove Theorem 2.4 by contradiction. Using the assumption that the equations in λ of the form $f_\lambda^m(\alpha) = \beta$ (as we vary m) have only finitely many solutions, we obtain that a certain plane curve (see equation (7.2.4)) contains *infinitely* many points from a suitable finitely generated subgroup of $\mathbb{G}_m^2$. This allows us to apply the main result of Moosa-Scanlon [MS04] on the structure of the intersection between a subvariety of a torus (in characteristic p) with a finitely generated subgroup to derive a contradiction.

Remark 2.5. We emphasize that both Theorems 2.4 and 2.2 also hold when $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$, i.e., the converse implication in Theorem 1.1 does not depend on whether all three points $\alpha_1, \alpha_2, \beta$ are in $\overline{\mathbb{F}}_p$.

2.3. **A general conjecture.** It is natural to consider a general colliding orbits problem for arbitrary families of polynomials in normal form. A polynomial of degree $d \geq 2$ is in *normal form* if it is monic and its coefficient for the monomial x^{d-1} is 0. In Conjecture 2.6, we allow both the starting points α_1, α_2 and the target point β to vary in an algebraic family as well.

Conjecture 2.6. *Let L be a field of characteristic p, and let $\alpha_1(z), \alpha_2(z), \beta(z) \in L[z]$. Suppose $f_\lambda(x) \in L[x]$ is a family of polynomials of degree $d \geq 2$ (parameterized by $\lambda \in \overline{L}$) in normal form, i.e.*

$$f_\lambda(x) = x^d + \sum_{i=0}^{d-2} c_i(\lambda) \cdot x^i, \tag{2.3.1}$$

for some polynomials $c_i(z) \in L[z]$ for $i = 0, \ldots, d-2$. We let

$$C(\alpha_1, \alpha_2; \beta) = \left\{\lambda \in \overline{L} \colon \ f_\lambda^m(\alpha_1(\lambda)) = \beta(\lambda) \text{ and } f_\lambda^n(\alpha_2(\lambda)) = \beta(\lambda) \text{ for some } m, n \in \mathbb{N}\right\}.$$

If $C(\alpha_1, \alpha_2; \beta)$ is infinite, then at least one of the following conditions must hold:

(1) *there exists a family of polynomials $g_\lambda(x)$ (similar to (2.3.1), but not necessarily normalized) and there exist integers $k > 0$ and $m, n \geq 0$ such that*

$$f_\lambda^k \circ g_\lambda = g_\lambda \circ f_\lambda^k \text{ and } f_\lambda^m(\alpha_1(\lambda)) = g_\lambda\left(f_\lambda^n(\alpha_2(\lambda))\right) \text{ (or } f_\lambda^m(\alpha_2(\lambda)) = g_\lambda\left(f_\lambda^n(\alpha_1(\lambda))\right)\text{),} \tag{2.3.2}$$

for all $\lambda \in \overline{L}$.

(2) *there exists $k \in \mathbb{N}$ such that for some $j \in \{1, 2\}$, we have that $f_\lambda^k(\alpha_j(\lambda)) = \beta(\lambda)$ for* all *$\lambda \in \overline{L}$.*

(3) *for* each *$\lambda \in \overline{L}$, the polynomial $\tilde{f}_\lambda(x) := f_\lambda(x) - c_0(\lambda)$ is additive (i.e., $\tilde{f}_\lambda(x+y) = \tilde{f}_\lambda(x) + \tilde{f}_\lambda(y)$ for all x, y). Furthermore, for each $\lambda \in \overline{L}$, we have that $\delta_1(\lambda) := \alpha_2(\lambda) - \alpha_1(\lambda)$ and $\delta_2(\lambda) := \beta(\lambda) - \alpha_1(\lambda)$ are preperiodic under the action of $\tilde{f}_\lambda(x)$.*

(4) *$c_i(z) \in \overline{\mathbb{F}}_p[z]$ for $i = 0, \ldots, d-2$ and $\alpha_1(z), \alpha_2(z), \beta(z) \in \overline{\mathbb{F}}_p[z]$.*

Remark 2.7. One could formulate Conjecture 2.6 for an arbitrary (unnormalized) family of polynomials $f_\lambda \in L[\lambda][x]$ (of degree $d \geq 2$), but this would complicate condition (4). When $p \nmid d$, this simplification has no loss of generality: any such family can be normalized through a linear conjugation, at the expense of parameterizing λ by a curve rather than the affine line.

We briefly discuss the statements (1)-(4) in the conclusion of Conjecture 2.6. Statement (1) is a significant generalization of conclusion (A) in Theorem 1.1; for arbitrary families of polynomials (2.3.1), the starting points $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ may be dynamically related through the much more complicated relation (2.3.2) from Conjecture 2.6 (see also [BD13, Theorem 1.3]). It is likely that to obtain a converse statement in Conjecture 2.6, i.e., that the set $C(\alpha_1, \alpha_2; \beta)$ is infinite, one would need a stronger statement than (1).

Statement (2) does not appear for the dynamical system considered in Theorem 1.1. However, for general dynamical systems, one needs to account for the possibility that $\beta(\lambda)$ is in the forward orbit of $\alpha_1(\lambda)$ or $\alpha_2(\lambda)$ (for *all* $\lambda \in \overline{L}$). If this were to happen, one would expect the corresponding set $C(\alpha_1, \alpha_2; \beta)$ to be infinite due to a possible extension of our Theorem 2.4.

Statement (3) asks that the only monomials x^i in $f_\lambda(x)$ (for $i > 0$) appearing with a nonzero coefficient correspond to $i = p^j$ for some $j \geq 0$; this is the generalization of conclusion (B) appearing in Theorem 1.1. Again, a converse to Conjecture 2.6 would require a more refined version of statement (3) (see the system (1.3.2) from Theorem 1.1 for $f_\lambda(z) = z^d + \lambda$).

Finally, we expect statement (4) from Conjecture 2.6 yields that $C(\alpha_1, \alpha_2; \beta)$ is infinite as long as the family $f_\lambda(z)$ does *not* satisfy statement (3) (similar to Conjecture 9.1 for the special family of polynomials $f_\lambda(z) = z^d + \lambda$) *and* also assuming that neither α_1 nor α_2 is *persistently preperiodic* for our family of polynomials (i.e., for $j = 1, 2$, there exists no $0 \leq m < n$ such that $f_\lambda^m(\alpha_j(\lambda)) = f_\lambda^n(\alpha_j(\lambda))$ for *all* $\lambda \in \overline{L}$). In Subsection 9.4, we present some numerical evidence supporting our expectation.

We expect Conjecture 2.6 to be *very difficult*. The main obstacle is the lack of a generalization of Theorem 6.9 (obtained in [Ghi] for $f_\lambda(z) := z^d + \lambda$) to arbitrary families of polynomials. This difficulty, in turn, stems from the fact that some of the key tools available in characteristic 0 for determining the *exact dynamical relation* between points of equal canonical height (with respect to the given family of polynomials) do not exist in characteristic p.

Remark 2.8. We expect that the analogue of Conjecture 2.6 for dynamical systems over a field L of characteristic 0 is more manageable and would only lead to the conclusions (1) and (2). The general strategy would follow the steps from our paper. While there would be no complications from additive polynomials (which are nontrivial only in characteristic p), new technical challenges would arise from dealing with arbitrary polynomial families. A potential line of attack would be to use the variation of the canonical height in families, as proven by Ingram [Ing13], to generalize (2.1.2) when $C(\alpha_1, \alpha_2; \beta)$ is infinite. Then using [BD13, Theorem 1.3], together with the description of the periodic plane curves under the coordinatewise action of polynomials from [MS14], one should be able to derive the conclusions (1)-(2) from Conjecture 2.6. There are further complications when the dynamical system $(f_\lambda, \alpha_1(\lambda), \alpha_2(\lambda), \beta(\lambda))$ is not defined over $\overline{\mathbb{Q}}$. However, we believe these can be overcome using the methods from [GHT13, Section 10] and the description of points of canonical height 0 from [Ben05] for a polynomial defined over a function field. We hope to return to this general question for dynamical systems over fields of characteristic 0 in a sequel paper.

3. Preliminary results

Let L be a field of characteristic p with a fixed algebraic closure $\overline{L}$. We consider the family of polynomials $f_\lambda(z) = z^d + \lambda$ parameterized by $\lambda \in \overline{L}$.

This Section is divided into two parts. In Subsection 3.1, we gather some information about the iterates of $f_\lambda(z)$. In Subsection 3.2, we work under the hypotheses of Theorem 2.1 for the points $\alpha_1, \alpha_2, \beta \in L$ (i.e., assuming the set $C(\alpha_1, \alpha_2; \beta)$ from (2.1.1) is infinite) to obtain information regarding the transcendence degree of $\mathbb{F}_p(\alpha_1, \alpha_2, \beta)/\mathbb{F}_p$.

3.1. **Iterates of our family of polynomials.** Let $\alpha \in L$. Following [Ghi, Subsection 3.1], for each $n \in \mathbb{N}$, there exists a polynomial $P_{n,\alpha}(\lambda) \in L[\lambda]$ of degree d^{n-1} such that

$$P_{n,\alpha}(\lambda) = f_\lambda^n(\alpha) \text{ for each } \lambda \in \overline{L}. \tag{3.1.1}$$

Remark 3.1. Using the notation from (3.1.1), we can reformulate the hypotheses from Theorem 1.1 (or from Theorem 2.1) that $C(\alpha_1, \alpha_2; \beta)$ is infinite as follows. Given $\alpha_1, \alpha_2, \beta \in L$, we are asking that there exist infinitely many $\lambda \in \overline{L}$ such that for some $m, n \in \mathbb{N}$,

$$P_{m,\alpha_1}(\lambda) - \beta = P_{n,\alpha_2}(\lambda) - \beta = 0;$$

or alternatively, the set of roots for all polynomials $\gcd\left(P_{m,\alpha_1}(x) - \beta, P_{n,\alpha_2}(x) - \beta\right) \in L[x]$ (as we vary $m, n \in \mathbb{N}$) is infinite.

A simple induction on n yields the following result.

Lemma 3.2. *For each $n \in \mathbb{N}$, the coefficients of the polynomial $P_{n,\alpha}(\lambda)$ are themselves polynomials in α, that is,*

$$P_{n,\alpha}(\lambda) = \sum_{i=0}^{d^{n-1}} c_{n,i}(\alpha)\lambda^{d^{n-1}-i}. \tag{3.1.2}$$

Furthermore, we have the following more precise information:

(a) $c_{n,0}(\alpha) = 1$;
(b) $c_{n,d^{n-1}}(\alpha) = \alpha^{d^n}$;
(c) $\deg_\alpha(c_{n,i}(\alpha)) \leq d \cdot i$, *for* $i = 0, \ldots, d^{n-1}$.

Proof. We prove that statements (a)-(c) hold by induction on n; the case $n = 1$ is obvious since $f_\lambda(\alpha) = \alpha^d + \lambda$ and so, $c_{1,0}(\alpha) = 1$, while $c_{1,1}(\alpha) = \alpha^d$.

Now, we assume the statements (a)-(c) hold for $c_{n,i}$ (for $0 \leq i \leq d^{n-1}$) and we prove the same statements also hold for $c_{n+1,i}$ and $0 \leq i \leq d^n$. We have

$$P_{n+1,\alpha}(\lambda) = P_{n,\alpha}(\lambda)^d + \lambda = \left(\sum_{i=0}^{d^{n-1}} c_{n,i}(\alpha)\lambda^{d^{n-1}-i}\right)^d + \lambda. \tag{3.1.3}$$

By inspecting the expansion, $c_{n+1,0}(\alpha) = c_{n,0}(\alpha)^d$ and $c_{n+1,d^n}(\alpha) = c_{n,d^{n-1}}(\alpha)^d$. Hence, statements (a)-(b) follow by the inductive hypothesis.

Finally, regarding statement (c), we assign weight d to λ and weight 1 to α. The total weight of each monomial (in α and λ) from $P_{n,\alpha}(\lambda)$ is therefore at most d^n (using the inductive hypothesis for (c) along with equation (3.1.2)). Then using the recurrence relation (3.1.3), we conclude that each monomial in $P_{n+1,\alpha}(\lambda)$ has weight at most d^{n+1}; therefore, the degree of each $c_{n+1,i}(\alpha)$ is at most $d \cdot i$, as desired. □

The following result is an easy consequence of Lemma 3.2.

Lemma 3.3. *For each $\alpha, \beta \in L$ and for each $n \in \mathbb{N}$, there exist finitely many $\lambda \in \overline{L}$ such that $f_\lambda^n(\alpha) = \beta$.*

Proof. Since $P_{n,\alpha}(\lambda) = \beta$ (see (3.1.2)) is an equation of degree d^{n-1} (in λ), there are only finitely many solutions $\lambda \in \overline{L}$. □

3.2. **Transcendence degree for the field generated by our points.** We let $\alpha_1, \alpha_2, \beta \in L$ and let $C(\alpha_1, \alpha_2; \beta)$ be the subset of $\overline{L}$ (as in (2.1.1)) consisting of all λ for which there exist some $m, n \in \mathbb{N}$ such that

$$f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta. \tag{3.2.1}$$

Lemma 3.4. *If $\lambda \in C(\alpha_1, \alpha_2; \beta)$, then $\lambda \in \overline{\mathbb{F}_p(\alpha_1, \beta)} \bigcap \overline{\mathbb{F}_p(\alpha_2, \beta)}$.*

Proof. Writing $f_\lambda^m(\alpha_1) = \beta$ as $P_{m,\alpha_1}(\lambda) = \beta$, i.e.,

$$\lambda^{d^{m-1}} + \sum_{i=0}^{d^{m-1}} c_{m,i}(\alpha_1)\lambda^{d^{m-1}-1} = \beta,$$

we obtain $\lambda \in \overline{\mathbb{F}_p(\alpha_1, \beta)}$. By symmetry, we also have $\lambda \in \overline{\mathbb{F}_p(\alpha_2, \beta)}$. □

Lemma 3.5. *If $C(\alpha_1, \alpha_2; \beta)$ is nonempty, then $\alpha_1 \in \overline{\mathbb{F}_p(\alpha_2, \beta)}$ and $\alpha_2 \in \overline{\mathbb{F}_p(\alpha_1, \beta)}$.*

Proof. Let $\lambda \in C(\alpha_1, \alpha_2; \beta)$, i.e., λ satisfies equations (3.2.1). Then Lemma 3.4 yields that

$$\lambda \in \overline{\mathbb{F}_p(\alpha_1, \beta)}. \tag{3.2.2}$$

Writing $f_\lambda^n(\alpha_2) = \beta$ as $P_{n,\alpha_2}(\lambda) = \beta$, i.e.,

$$\lambda^{d^{n-1}} + c_{n,1}(\alpha_2)\lambda^{d^{n-1}-1} + \cdots + c_{n,d^{n-1}-1}(\alpha_2)\lambda + \alpha_2^{d^n} = \beta, \tag{3.2.3}$$

where for each $i = 1, \ldots, d^{n-1} - 1$, $c_{n,i}(\alpha_2)$ is a polynomial in α_2 of degree at most $d \cdot i < d^n$ (according to Lemma 3.2 (c)), we conclude that $\alpha_2 \in \overline{\mathbb{F}_p(\lambda, \beta)}$. Then equation (3.2.2) yields the desired conclusion that $\alpha_2 \in \overline{\mathbb{F}_p(\alpha_1, \beta)}$. By symmetry, we also have $\alpha_1 \in \overline{\mathbb{F}_p(\alpha_2, \beta)}$. □

Lemma 3.6. *If $C(\alpha_1, \alpha_2; \beta)$ is infinite, then $\alpha_1 \in \overline{\mathbb{F}_p(\alpha_2)}$ and $\alpha_2 \in \overline{\mathbb{F}_p(\alpha_1)}$.*

Proof. Suppose $C(\alpha_1, \alpha_2; \beta)$ is infinite. Due to the symmetry between α_1 and α_2, it suffices to prove that $\alpha_1 \in \overline{\mathbb{F}_p(\alpha_2)}$ (since an identical argument would yield $\alpha_2 \in \overline{\mathbb{F}_p(\alpha_1)}$).

We let $L_1 := \overline{\mathbb{F}_p(\alpha_2)}$ and $K := L_1(\alpha_1, \beta)$. If $K = L_1$, then $\alpha_1 \in L_1 = \overline{\mathbb{F}_p(\alpha_2)}$, as desired. Henceforth, we assume K/L_1 is a function field, either of transcendence degree equal to 1, or of transcendence degree equal to 2 (in which case, α_1 and β are algebraically independent over L_1). We argue by contradiction and assume that $\alpha_1 \notin L_1$.

Let V be a smooth projective variety defined over L_1 whose function field equals K (either V is a curve if $\mathrm{trdeg}_{L_1} K = 1$, or $V = \mathbb{P}^2$ if α_2 and β are algebraically independent over L_1). Let Ω_V be an inequivalent set of absolute values on K corresponding to the irreducible divisors of V. For the function field K/L_1, we have:

$$|\gamma|_w \leq 1 \text{ for each } w \in \Omega_V \text{ if and only if } \gamma \in L_1. \tag{3.2.4}$$

Since $\alpha_1 \notin L_1$, there exists $v \in \Omega_V$ such that $|\alpha_1|_v > 1$. We also fix an extension of $|\cdot|_v$ to an absolute value on $\overline{K} \subseteq \overline{L}$.

Let $\lambda \in C(\alpha_1, \alpha_2; \beta)$; so, there exist $m, n \in \mathbb{N}$ such that $f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta$. Since we assumed that $C(\alpha_1, \alpha_2; \beta)$ is infinite, Lemma 3.3 allows us to assume that both m and n are arbitrarily large; in particular, we may assume that

$$|\beta|_v < |\alpha_1|_v^{d^{\min\{m,n\}}}. \tag{3.2.5}$$

The equation $f_\lambda^n(\alpha_2) = \beta$ yields $P_{n,\alpha_2}(\lambda) = \beta$ and so, using equation (3.1.2) along with the fact that $|\alpha_2|_v \leq 1$ (see (3.2.4)), we conclude that

$$|\lambda|_v \leq \max\{|\beta|_v, 1\}^{\frac{1}{d^{n-1}}}. \tag{3.2.6}$$

On the other hand, the equation $f_\lambda^m(\alpha_1) = \beta$ yields $P_{m,\alpha_1}(\lambda) = \beta$, i.e.,

$$\lambda^{d^{m-1}} + \sum_{i=1}^{d^{m-1}} c_{m,i}(\alpha_1) \cdot \lambda^{d^{m-1}-i} = \beta. \tag{3.2.7}$$

Lemma 3.2 (c) yields that $\deg(c_{m,i}) \leq d \cdot i$ for each $i = 1, \ldots, d^{m-1}$; also, $c_{m,d^{m-1}}(\alpha_1) = \alpha_1^{d^m}$. Since $|\beta|_v < |\alpha_1|_v^{d^m}$ by (3.2.5), there exists some $i \in \{0, \ldots, d^{m-1} - 1\}$ such that

$$\left| c_{m,i}(\alpha) \cdot \lambda^{d^{m-1}-i} \right|_v \geq |\alpha_1|_v^{d^m}. \tag{3.2.8}$$

Because each $c_{m,i}$ is a polynomial with coefficients in $\mathbb{F}_p$ of degree at most $d \cdot i$ (according to Lemma 3.2 (c)), we have that

$$|c_{m,i}(\alpha_1)|_v \leq |\alpha_1|_v^{d \cdot i} \text{ for each } i = 0, \ldots, d^{m-1} - 1. \tag{3.2.9}$$

Combining inequalities (3.2.8) and (3.2.9), we obtain that

$$|\lambda|_v \geq |\alpha_1|_v^d. \tag{3.2.10}$$

Next, combining inequalities (3.2.6) and (3.2.10), we get

$$\max\{|\beta|_v, 1\} \geq |\lambda|_v^{d^{n-1}} \geq |\alpha_1|_v^{d^n} > 1. \tag{3.2.11}$$

Inequalities (3.2.5) and (3.2.11) provide a contradiction; so, we must have that $\alpha_1 \in L_1$.

This concludes our proof of Lemma 3.6. □

Proposition 3.7. *Let $\alpha_1, \alpha_2, \beta \in L$ and assume $C(\alpha_1, \alpha_2; \beta)$ is infinite. Then at least one of the following two conditions must hold:*

(1) $\alpha_1^d = \alpha_2^d$.
(2) $\beta \in \overline{\mathbb{F}_p(\alpha_1)} = \overline{\mathbb{F}_p(\alpha_2)}$.

Proof. First, Lemma 3.6 yields the equality $\overline{\mathbb{F}_p(\alpha_1)} = \overline{\mathbb{F}_p(\alpha_2)}$. So, letting $L_1 := \overline{\mathbb{F}_p(\alpha_1)}$, it suffices to prove that if $\beta \notin L_1$, then $\alpha_1^d = \alpha_2^d$.

We let $K_1 = L_1(\beta)$; this is a rational function field of transcendence degree 1 (since we assumed that $\beta \notin L_1$). We view K_1 as the function field of $\mathbb{P}^1$ over L_1. Let $|\cdot|_\infty$ be the absolute value on the function field K_1/L_1 corresponding to the place at infinity from $\mathbb{P}^1_{L_1}$; hence $|\beta|_\infty > 1$. We fix an extension of $|\cdot|_\infty$ to an absolute value on the algebraic closure $\overline{K_1}$.

Let $j \in \{1, 2\}$, let $\lambda \in \overline{K_1}$ and let $\ell \in \mathbb{N}$ such that $f_\lambda^\ell(\alpha_j) = \beta$. Then equation (3.1.2) yields

$$\lambda^{d^{\ell-1}} + \sum_{i=1}^{d^{\ell-1}} c_{\ell,i}(\alpha_j) \cdot \lambda^{d^{\ell-1}-i} = \beta. \tag{3.2.12}$$

Since $|c_{\ell,i}(\alpha_j)|_\infty \leq 1$ (note that $\alpha_j \in L_1$), we get that

$$|\lambda|_\infty = |\beta|_\infty^{\frac{1}{d^{\ell-1}}} > 1. \tag{3.2.13}$$

Now, let $\lambda \in C(\alpha_1, \alpha_2; \beta)$, and let $m, n \in \mathbb{N}$ such that $f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta$. Equation (3.2.13) yields that

$$|\lambda|_\infty = |\beta|_\infty^{\frac{1}{d^{m-1}}} = |\beta|_\infty^{\frac{1}{d^{n-1}}};$$

since $|\beta|_\infty > 1$, we have $m = n$. Consequently, $P_{m,\alpha_1}(\lambda) = P_{m,\alpha_2}(\lambda) = \beta$, i.e.

$$\lambda^{d^{m-1}} + \sum_{i=1}^{d^{m-1}} c_{m,i}(\alpha_1) \cdot \lambda^{d^{m-1}-i} = \lambda^{d^{m-1}} + \sum_{i=1}^{d^{m-1}} c_{m,i}(\alpha_2) \cdot \lambda^{d^{m-1}-i}, \tag{3.2.14}$$

with the notation for $c_{m,i}$ as in Lemma 3.2. Noting that $c_{m,d^m}(\alpha_j) = \alpha_j^{d^m}$ for $j = 1, 2$, equation (3.2.14) yields

$$\sum_{i=1}^{d^{m-1}-1} (c_{m,i}(\alpha_1) - c_{m,i}(\alpha_2)) \cdot \lambda^{d^{m-1}-i} + \left(\alpha_1^{d^m} - \alpha_2^{d^m}\right) = 0. \tag{3.2.15}$$

Because λ is transcendental over L_1 (due to (3.2.13)), equation (3.2.15) yields that

$$\alpha_1^{d^m} = \alpha_2^{d^m} \text{ and} \tag{3.2.16}$$

$$c_{m,i}(\alpha_1) = c_{m,i}(\alpha_2) \text{ for each } i = 1, \ldots, d^{m-1} - 1. \tag{3.2.17}$$

Now, we write $d = p^r \cdot s$, for some nonnegative integer r and some positive integer s, which is not divisible by p. If $s = 1$, i.e., $d = p^r$, then equation (3.2.16) yields that $\alpha_1 = \alpha_2$ and therefore, condition (1) from the conclusion of Proposition 3.7 holds.

Next, we assume $s \geq 2$.

Lemma 3.8. *With the above notation, we have that $c_{m,i} = 0$ for $i = 1, \ldots, p^{(m-1)r} - 1$, while*

$$c_{m,p^{(m-1)r}}(\alpha_j) = s^{(p^{r(m-1)}-1)/(p^r-1)} \alpha_j^{p^{rm}s} \text{ for } j = 1, 2.$$

Proof of Lemma 3.8. The result follows readily by induction on m, using the recurrence relation (3.1.3). The desired formula holds trivially also when $m = 1$. To aid the reader's intuition for the general case, we show the computation for $m = 2$. For the sake of simplifying our notation in Lemma 3.8), we let $\alpha := \alpha_j$ (for $j = 1, 2$) and observe that

$$P_{2,\alpha}(\lambda) = \left(\alpha^d + \lambda\right)^d = \left(\alpha^{sp^r} + \lambda\right)^{p^r \cdot s} + \lambda = \left(\alpha^{sp^{2r}} + \lambda^{p^r}\right)^s + \lambda,$$

which yields that

$$P_{2,\alpha}(\lambda) = \lambda^{p^r \cdot s} + s \cdot \alpha^{sp^{2r}} \cdot \lambda^{p^r(s-1)} + \text{ lower order terms},$$

proving the claimed formula when $m = 2$. For the inductive step, assume the formula holds for some $m \geq 1$:

$$P_{m,\alpha}(\lambda) = \lambda^{p^{(m-1)r} \cdot s^{m-1}} + s^{(p^{r(m-1)}-1)/(p^r-1)} \alpha^{p^{rm}s} \cdot \lambda^{p^{(m-1)r} \cdot \left(s^{m-1}-1\right)} + \text{ lower order terms}.$$

Then the recurrence relation (3.1.3) yields

$$P_{m+1,\alpha}(\lambda) = \left(\lambda^{p^{(m-1)r} \cdot s^{m-1}} + s^{(p^{r(m-1)}-1)/(p^r-1)} \alpha^{p^{rm}s} \cdot \lambda^{p^{(m-1)r} \cdot \left(s^{m-1}-1\right)} + \cdots\right)^{p^r \cdot s} + \lambda.$$

Since the field has characteristic p, we can distribute the p^r exponent, which gives

$$P_{m+1,\alpha}(\lambda) = \left(\lambda^{p^{mr}\cdot s^{m-1}} + s^{p^r\cdot(p^{r(m-1)}-1)/(p^r-1)}\alpha^{p^{r(m+1)}s}\cdot\lambda^{p^{mr}\cdot\left(s^{m-1}-1\right)} + \cdots\right)^s + \lambda.$$

We conclude that

$$P_{m+1,\alpha}(\lambda) = \lambda^{p^{mr}\cdot s^m} + s\cdot s^{p^r\cdot(p^{r(m-1)}-1)/(p^r-1)}\alpha^{p^{r(m+1)}s}\cdot\lambda^{p^{mr}\cdot s^{m-1}\cdot(s-1)+p^{mr}\cdot\left(s^{m-1}-1\right)} + \cdots, \text{ i.e.,}$$

$$P_{m+1,\alpha}(\lambda) = \lambda^{d^m} + s^{\frac{p^{rm}-1}{p^r-1}}\alpha^{p^{r(m+1)}s}\cdot\lambda^{d^m-p^{mr}} + \text{ lower order terms,}$$

which completes the inductive step and thus proves Lemma 3.8. □

Applying Lemma 3.8, the equation (3.2.17) implies that:

$$s^{(p^{r(m-1)}-1)/(p^r-1)}\alpha_1^{p^{rm}s} = s^{(p^{r(m-1)}-1)/(p^r-1)}\alpha_2^{p^{rm}s}. \tag{3.2.18}$$

Since $s\neq 0$ in $\mathbb{F}_p$, equation (3.2.18) yields $\alpha_1^s=\alpha_2^s$ and therefore, $\alpha_1^d=\alpha_2^d$, as desired.

This concludes our proof of Proposition 3.7. □

4. Heights

In this Section, we establish the framework of absolute values and heights needed for our proof of Theorem 2.1. Throughout Section 4, we let t be a transcendental element over $\overline{\mathbb{F}}_p$.

4.1. **Absolute values for the one-variable rational function field.** We let $\Omega_0 := \Omega_{\overline{\mathbb{F}}_p(t)}$ be the set of absolute values on $\overline{\mathbb{F}}_p(t)$ defined as follows. For each $c\in\overline{\mathbb{F}}_p$, we have the unique absolute value $|\cdot|_{v_c}$ in Ω_0 normalized as follows: for any nonzero rational function $\frac{g_1}{g_2}$ (with $g_1,g_2\in\overline{\mathbb{F}}_p[t]\setminus\{0\}$), we have

$$\left|\frac{g_1}{g_2}\right|_{v_c} := e^{\mathrm{ord}_c(g_2)-\mathrm{ord}_c(g_1)},$$

where $\mathrm{ord}_c(g)$ is the order of vanishing at the point $c\in\overline{\mathbb{F}}_p$ of any nonzero polynomial $g\in\overline{\mathbb{F}}_p[t]$. Besides the above absolute values $|\cdot|_{v_c}$, we also have the absolute value $|\cdot|_{v_\infty}\in\Omega_0$ normalized as follows: for any nonzero rational function $\frac{g_1}{g_2}$ (with $g_1,g_2\in\overline{\mathbb{F}}_p[t]\setminus\{0\}$), we have

$$\left|\frac{g_1}{g_2}\right|_{v_\infty} = e^{\deg(g_1)-\deg(g_2)}.$$

It is immediate that for each nonzero rational function $g\in\overline{\mathbb{F}}_p(t)$, we have the following product formula:

$$\prod_{v\in\Omega_0}|g|_v = 1. \tag{4.1.1}$$

4.2. **Extending the absolute values to the perfect closure.** We let $L_0\subset\overline{L}$ be the perfect closure of $\overline{\mathbb{F}}_p(t)$, i.e.,

$$L_0 := \overline{\mathbb{F}}_p\left(t, t^{1/p}, t^{1/p^2}, \cdots, t^{1/p^n}, \cdots\right). \tag{4.2.1}$$

The field L_0 is *perfect*, meaning all its finite extensions are separable.

Each absolute value $|\cdot|_v$ from Ω_0 has a unique extension to an absolute value on L_0; we denote by Ω_{L_0} the set of all these extended absolute values. Once again, we have a product formula for each nonzero $\gamma \in L_0$:

$$\prod_{v\in\Omega_{L_0}} |\gamma|_v = 1. \tag{4.2.2}$$

4.3. **Heights for any algebraic element.** For any real number u, we define $\log^+|u| := \log\max\{|u|, 1\}$.

We fix a finite extension K of L_0. For any absolute value $|\cdot|_v \in \Omega_{L_0}$, there exist finitely many places w of K lying above the place v of L_0, denoted by $w|v$. We normalize the corresponding absolute values $|\cdot|_w$ on K, and we denote by $\Omega := \Omega_K$ the corresponding set of absolute values. For each $\gamma \in L_0$ and each $v \in \Omega_{L_0}$, we have the following relation:

$$|\gamma|_v = \prod_{\substack{w\in\Omega_K \\ w|v}} |\gamma|_w. \tag{4.3.1}$$

Using equations (4.3.1) and (4.2.2), K satisfies the product formula with respect to the absolute values from $\Omega = \Omega_K$, i.e.,

$$\prod_{w\in\Omega} |\gamma|_w = 1 \text{ for each nonzero } \gamma \in K.$$

We fix an algebraic closure $\overline{K}$ of K. Then for each $w \in \Omega = \Omega_K$, we fix an extension of $|\cdot|_w$ to an absolute value on $\overline{K}$.

For any $\gamma \in \overline{K}$, we define the *Weil height* of γ as follows:

$$h(\gamma) := h_K(\gamma) := \frac{1}{[K(\gamma):K]} \cdot \sum_{\substack{\sigma:K(\gamma)\to\overline{K} \\ \sigma|_K=\mathrm{id}_K}} \sum_{v\in\Omega} \log^+ |\sigma(\gamma)|_v\,. \tag{4.3.2}$$

Remark 4.1. By construction, the Weil height depends on our choice of field K. In our proof of Theorem 2.1, we will choose a field K so that it contains $\alpha_1, \alpha_2, \beta$ (for more details, see Section 6).

4.4. **Canonical heights.** For any $\lambda \in \overline{K}$, we consider the polynomial $f_\lambda(z) = z^d + \lambda$ of degree $d \geq 2$. Let $\gamma \in \overline{K}$. For each $v \in \Omega = \Omega_K$, we construct the *local canonical height*:

$$\widehat{h}_{\lambda,v}(\gamma) := \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\gamma)|_v}{d^n}. \tag{4.4.1}$$

Equation (4.4.1) yields that for each $m \in \mathbb{N}$:

$$\widehat{h}_{\lambda,v}\left(f_\lambda^m(\gamma)\right) = d^m \cdot \widehat{h}_{\lambda,v}(\gamma). \tag{4.4.2}$$

We define the *(global) canonical height* $\widehat{h}_\lambda(\gamma)$ as follows:

$$\widehat{h}_\lambda(\alpha) = \lim_{n\to\infty} \frac{h\left(f_\lambda^n(\gamma)\right)}{d^n}. \tag{4.4.3}$$

Equation (4.4.3) yields that for each $m \in \mathbb{N}$:

$$\widehat{h}_\lambda\left(f_\lambda^m(\gamma)\right) = d^m \cdot \widehat{h}_\lambda(\gamma). \tag{4.4.4}$$

For more details regarding the canonical height associated to a polynomial, see [Sil07, Section 3.4] and also [CS93]. Furthermore, we have the following connection between the local and the global canonical heights.

Lemma 4.2. *Let $\alpha \in K$ and $\lambda \in \overline{K}$. Then the following holds:*

$$\widehat{h}_\lambda(\alpha) = \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega_K} \widehat{h}_{\sigma(\lambda),v}(\alpha). \tag{4.4.5}$$

Proof. Employing equations (4.4.3), (4.4.1) and (4.3.2), we have:

$$\widehat{h}_\lambda(\alpha) = \frac{1}{[K(\lambda):K]} \cdot \lim_{n\to\infty} \frac{1}{d^n} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega_K} \log^+ \left|\sigma\left(f_\lambda^n(\alpha)\right)\right|_v \tag{4.4.6}$$

and so, because $\alpha \in K$, we get

$$\widehat{h}_\lambda(\alpha) = \frac{1}{[K(\lambda):K]} \cdot \lim_{n\to\infty} \frac{1}{d^n} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega_K} \log^+ \left|f_{\sigma(\lambda)}^n(\alpha)\right|_v ; \text{ therefore,} \tag{4.4.7}$$

$$\widehat{h}_\lambda(\alpha) = \frac{1}{[K(\lambda):K]} \cdot \sum_{v \in \Omega_K} \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \widehat{h}_{\sigma(\lambda),v}(\alpha). \tag{4.4.8}$$

The limit and the corresponding summation over all $v \in \Omega_K$ in (4.4.7) can be interchanged, because the limit equals 0 for all but finitely many places v (see [Ghi, Lemma 3.4]). □

5. Bounds for the canonical height for our family of polynomials

We continue with the framework for local and global canonical heights from Section 4. Let

$$L_0 = \overline{\mathbb{F}}_p\left(t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \cdots\right)$$

and let K be a finite extension of L_0. We denote by Ω_0 the set of absolute values on L_0 constructed as in Subsection 4.2. Also, we let $\Omega := \Omega_K$ be the normalized absolute values on K corresponding to the places of K lying above the places from Ω_0 (see Subsection 4.3). In addition, we fix an extension of each $|\cdot|_v$ to an absolute value on $\overline{K}$. We construct the local and global canonical heights (see equations (4.4.1) and (4.4.3)) with respect to polynomials $f_\lambda(z) := z^d + \gamma$ (for $\gamma \in \overline{K}$). The following result is proved in [Ghi, Lemma 3.4].

Lemma 5.1. *Let $v \in \Omega$ and let $\gamma, \lambda \in \overline{K}$.*

(i) If $|\gamma|_v \le 1$ and $|\lambda|_v \le 1$, then $\widehat{h}_{\lambda,v}(\gamma) = 0$.
(ii) If $|\gamma|_v^d > \max\{1, |\lambda|_v\}$, then $\widehat{h}_{\lambda,v}(\gamma) = \log|\gamma|_v > 0$.
(iii) If $|\lambda|_v > \max\{1, |\gamma|_v^d\}$, then $\widehat{h}_{\lambda,v}(\gamma) = \frac{\log|\lambda|_v}{d} > 0$.

Lemma 5.1 yields the following key inequality.

Proposition 5.2. *For $\alpha \in K$ and $\lambda \in \overline{K}$, we have that*

$$\frac{h(\lambda)}{d} - h(\alpha) \le \widehat{h}_\lambda(\alpha) \le \frac{h(\lambda)}{d} + h(\alpha). \tag{5.0.1}$$

Proposition 5.2 belongs to a long series of results regarding the variation of the canonical height in algebraic families. Most of these results are formulated over number fields (see [CS93, DM23, GM13, Ing13, Sil92, Sil94a, Sil94b, Tat83]); by contrast, only a few such results are stated over function fields of positive characteristic (see [Ghi24]).

Proof of Proposition 5.2. We first prove the right-hand side inequality from (5.0.1).

Let $v \in \Omega = \Omega_K$ and let $\sigma\colon K(\gamma) \to \overline{K}$ be a field homomorphism fixing K pointwise. We let

$$M_{v,\sigma} := \log^+ |\alpha|_v + \frac{\log^+ |\sigma(\lambda)|_v}{d}. \tag{5.0.2}$$

Lemma 5.1 (i) shows that

$$\text{if } |\alpha|_v, |\sigma(\lambda)|_v \leq 1, \text{ then } \widehat{h}_{\sigma(\lambda),v}(\alpha) = 0 = M_{v,\sigma}. \tag{5.0.3}$$

Next, assume that $\max\{|\alpha|_v, |\sigma(\lambda)|_v\} > 1$, i.e., $M_{v,\sigma} > 0$. If $|\alpha|_v > |\sigma(\lambda)|_v^{\frac{1}{d}}$, then Lemma 5.1 (ii) yields that

$$\widehat{h}_{\sigma(\lambda),v}(\alpha) = \log |\alpha|_v \leq M_{v,\sigma}. \tag{5.0.4}$$

If $|\alpha|_v < |\sigma(\lambda)|^{\frac{1}{d}}$, then Lemma 5.1 (iii) guarantees that

$$\widehat{h}_{\sigma(\lambda),v} = \frac{\log |\sigma(\lambda)|_v}{d} \leq M_{v,\sigma}. \tag{5.0.5}$$

Now, if $|\alpha|_v = |\sigma(\lambda)|_v^{\frac{1}{d}} > 1$, then we get $\left|f_{\sigma(\lambda)}(\alpha)\right|_v \leq |\alpha|_v^d = |\sigma(\lambda)|_v$ and so,

$$\log \left|f_{\sigma(\lambda)}(\alpha)\right|_v < d \cdot M_{v,\sigma}. \tag{5.0.6}$$

For each $n \geq 1$, we have

$$\left|f^{n+1}_{\sigma(\lambda)}(\alpha)\right|_v \leq \max\left\{\left|f^n_{\sigma(\lambda)}(\alpha)\right|_v^d, |\sigma(\lambda)|_v\right\}. \tag{5.0.7}$$

Employing inequalities (5.0.6) and (5.0.7), a simple induction on n results in:

$$\left|f^n_{\sigma(\lambda)}(\alpha)\right|_v \leq d^n \cdot M_{v,\sigma}. \tag{5.0.8}$$

Inequality (5.0.8) and the definition (4.4.1) of the local canonical height together give:

$$\widehat{h}_{\sigma(\lambda),v}(\alpha) \leq M_{v,\sigma}. \tag{5.0.9}$$

Using equations (5.0.9), (5.0.5), (5.0.3) and (5.0.4), along with Lemma 4.2, we get that

$$\widehat{h}_\lambda(\alpha) = \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma:K(\lambda)\to\overline{K}\\ \sigma|_K = \mathrm{id}|_K}} \sum_{v\in\Omega} \widehat{h}_{v,\sigma(\lambda)}(\alpha) \leq \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma:K(\lambda)\to\overline{K}\\ \sigma|_K = \mathrm{id}|_K}} \sum_{v\in\Omega} M_{v,\sigma} \text{ and so,}$$

$$\widehat{h}_\lambda(\alpha) \leq \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma:K(\lambda)\to\overline{K}\\ \sigma|_K = \mathrm{id}|_K}} \sum_{v\in\Omega} \left(\log^+ |\alpha|_v + \frac{\log^+ |\sigma(\lambda)|_v}{d}\right), \text{ which yields}$$

$$\widehat{h}_\lambda(\alpha) \leq \left(\sum_{v\in\Omega} \log^+ |\alpha|_v\right) + \frac{1}{d\cdot[K(\lambda):K]} \cdot \sum_{\substack{\sigma:K(\lambda)\to\overline{K}\\ \sigma|_K = \mathrm{id}|_K}} \sum_{v\in\Omega} \log^+ |\sigma(\lambda)|_v; \text{ hence}$$

(5.0.10) $$\widehat{h}_\lambda(\alpha) \leq h(\alpha) + \frac{h(\lambda)}{d}, \text{ thus proving the right-hand side of (5.0.1).}$$

Next, we will establish the inequality from the left-hand side of (5.0.1). Again, we obtain suitable inequalities for each place $v \in \Omega$; this time, we define $N_{v,\sigma} := \frac{\log^+ |\sigma(\lambda)|_v}{d} - \log^+ |\alpha|_v$. Immediately, we note that

(5.0.11) $$\text{if } \log^+ |\sigma(\lambda)|_v \leq d \cdot \log^+ |\alpha|_v, \text{ then } \widehat{h}_{\sigma(\lambda),v}(\alpha) \geq 0 \geq N_v.$$

On the other hand, if $|\sigma(\lambda)|_v > \max\left\{1, |\alpha|_v^d\right\}$, then Lemma 5.1 (iii) yields that

(5.0.12) $$\widehat{h}_{\sigma(\lambda),v}(\alpha) = \frac{\log |\sigma(\lambda)|_v}{d} = \frac{\log^+ |\sigma(\lambda)|_v}{d} \geq N_v.$$

Combining inequalities (5.0.11) and (5.0.12) along with Lemma 4.2, we obtain

$$\widehat{h}_\lambda(\alpha) = \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega} \widehat{h}_{v,\sigma(\lambda)}(\alpha) \geq \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega} N_{v,\sigma}$$

and so,

$$\widehat{h}_\lambda(\alpha) \geq \frac{1}{[K(\lambda):K]} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega} \left(\frac{\log^+ |\sigma(\lambda)|_v}{d} - \log^+ |\alpha|_v \right), \text{ which yields}$$

$$\widehat{h}_\lambda(\alpha) \geq \left(\frac{1}{d \cdot [K(\lambda):K]} \cdot \sum_{\substack{\sigma: K(\lambda) \to \overline{K} \\ \sigma|_K = \mathrm{id}|_K}} \sum_{v \in \Omega} \log^+ |\sigma(\lambda)|_v \right) - \left(\sum_{v \in \Omega} \log^+ |\alpha|_v \right); \text{ hence}$$

(5.0.13) $$\widehat{h}_\lambda(\alpha) \geq \frac{h(\lambda)}{d} - h(\alpha), \text{ as desired for the left-hand side of (5.0.1).}$$

Inequalities (5.0.10) and (5.0.13) establish the desired conclusion of Proposition 5.2. □

The following result is instrumental in our proof for Theorem 2.1.

Proposition 5.3. *Let $d \geq 2$ be an integer, let $\alpha, \beta \in K$ and let $\lambda \in \overline{K}$. Let $f_\lambda(z) := z^d + \lambda$ and let $m \in \mathbb{N}$. If $f_\lambda^m(\alpha) = \beta$, then*

(5.0.14) $$\widehat{h}_\lambda(\alpha) \leq \frac{2h(\alpha) + 2h(\beta)}{d^m}.$$

Proof. From Proposition 5.2, we have the inequalities:

(5.0.15) $$\widehat{h}_\lambda(\alpha) \geq \frac{h(\lambda)}{d} - h(\alpha) \text{ and } \widehat{h}_\lambda(\beta) \leq \frac{h(\lambda)}{d} + h(\beta).$$

Using the condition $f_\lambda^m(\alpha) = \beta$ and the identity $d^m \widehat{h}_\lambda(\alpha) = \widehat{h}_\lambda(\beta)$ from equation (4.4.4), we combine the inequalities in (5.0.15) to obtain:

(5.0.16) $$d^m \cdot \left(\frac{h(\lambda)}{d} - h(\alpha) \right) \leq \frac{h(\lambda)}{d} + h(\beta).$$

Rearranging the terms in (5.0.16) gives an upper bound on the height of λ:

$$h(\lambda) \leq \frac{d^m \cdot h(\alpha) + h(\beta)}{\frac{d^m-1}{d}}. \tag{5.0.17}$$

Using the fact that $d^m - 1 \geq \frac{d^m}{2}$ (since $d \geq 2$ and $m \geq 1$), inequality (5.0.17) becomes

$$h(\lambda) \leq \frac{2d^{m+1} \cdot h(\alpha) + 2d \cdot h(\beta)}{d^m}. \tag{5.0.18}$$

Combining inequality (5.0.18) with the second inequality in (5.0.15) leads to

$$\widehat{h}_\lambda(\beta) \leq \frac{h(\lambda)}{d} + h(\beta) \leq \frac{2d^{m+1} \cdot h(\alpha) + 2d \cdot h(\beta)}{d^{m+1}} + h(\beta). \tag{5.0.19}$$

Since $d^{m+1} \geq 2d$ for $d \geq 2$, inequality (5.0.19) implies

$$\widehat{h}_\lambda(\beta) \leq 2h(\alpha) + h(\beta) + h(\beta) = 2h(\alpha) + 2h(\beta). \tag{5.0.20}$$

Finally, using $\widehat{h}_\lambda(\alpha) = \frac{\widehat{h}_\lambda(\beta)}{d^m}$ from equation (4.4.4) along with inequality (5.0.20), we obtain

$$\widehat{h}_\lambda(\alpha) \leq \frac{2h(\alpha) + 2h(\beta)}{d^m},$$

as desired. □

6. Proof of Theorem 2.1

We work with the notation and hypotheses from Theorem 2.1. In particular, we have the family of polynomials $f_\lambda(z) = z^d + \lambda$ (with a fixed $d \geq 2$) parameterized by $\lambda \in \overline{L}$, where L is a field of characteristic p. Furthermore, for some $\alpha_1, \alpha_2, \beta \in L$, the set

$$C(\alpha_1, \alpha_2; \beta) = \left\{\lambda \in \overline{L} \colon \text{ there exist } m, n \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta\right\} \tag{6.0.1}$$

is assumed to be infinite. Our goal is to prove that one of the alternatives (i)-(iii) in Theorem 2.1 must hold. We split our analysis into several subsections.

6.1. Reduction to a field of transcendence degree one.

We first prove that we may assume α_1 is transcendental over $\mathbb{F}_p$.

Proposition 6.1. *Under the hypotheses from Theorem 2.1, at least one of the following three alternatives must hold:*

(A) $\alpha_1^d = \alpha_2^d$.
(B) α_1 *is transcendental over* $\mathbb{F}_p$, *while* $\alpha_2, \beta \in \overline{\mathbb{F}_p(\alpha_1)} \subseteq \overline{L}$.
(C) $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$.

Proof. Either α_1 belongs to $\overline{\mathbb{F}}_p$ or α_1 is is transcendental over $\mathbb{F}_p$. First, if $\alpha_1 \in \overline{\mathbb{F}}_p \subseteq \overline{L}$, then $\alpha_2 \in \overline{\mathbb{F}}_p$ by Lemma 3.6. Applying Proposition 3.7 yields $\alpha_1^d = \alpha_2^d$ which is alternative (A), or $\beta \in \overline{\mathbb{F}}_p$ which is alternative (C).

Next, assume α_1 is transcendental over $\mathbb{F}_p$. Lemma 3.6 yields $\alpha_2 \in \overline{\mathbb{F}_p(\alpha_1)}$. Furthermore, Proposition 3.7 yields $\beta \in \overline{\mathbb{F}_p(\alpha_1)}$ which is alternative (B), or $\alpha_1^d = \alpha_2^d$ which is alternative (A). The proof of Proposition 6.1 is complete in all cases. □

The alternatives (A) and (C) from Proposition 6.1 match the alternatives (i) and (iii) from the conclusion of Theorem 2.1. In light of Lemma 3.6 and Proposition 3.7, we may henceforth assume that:

$$\alpha_1 \notin \overline{\mathbb{F}}_p \text{ and also, } \alpha_2, \beta \in \overline{\mathbb{F}_p(\alpha_1)}. \tag{6.1.1}$$

We will prove that under hypothesis (6.1.1), at least one of the alternatives (i) or (ii) from Theorem 2.1 must hold, i.e.,

(i) $\alpha_1^d = \alpha_2^d$; or
(ii) d is a power of the characteristic p of our field L, and $\alpha_1 - \beta, \alpha_2 - \beta \in \overline{\mathbb{F}}_p$.

6.2. **Two sequences of heights tending to zero.** Working under the assumption (6.1.1), let $t := \alpha_1$ (which is transcendental over $\overline{\mathbb{F}}_p$) and also, let

$$L_0 := \overline{\mathbb{F}}_p\left(t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \cdots, t^{\frac{1}{p^k}}, \cdots\right).$$

Then, according to (6.1.1), there exists a finite extension K of L_0 such that $\alpha_1, \alpha_2, \beta \in K$. As a consequence of Lemma 3.4, we have that $C(\alpha_1, \alpha_2; \beta) \subset \overline{K}$. Next, we obtain an easy consequence of Proposition 5.3, which is key for our argument.

Proposition 6.2. *There exist an infinite subset $\{\lambda_k\}$ in $\overline{K}$ such that*

$$\lim_{k\to\infty} \widehat{h}_{\lambda_k}(\alpha_1) = \lim_{k\to\infty} \widehat{h}_{\lambda_k}(\alpha_2) = 0. \tag{6.2.1}$$

Proof. Using Lemma 3.3, for each $\ell \in \mathbb{N}$ there exist finitely many $\lambda \in \overline{K}$ such that $f_\lambda^\ell(\alpha_1) = \beta$ (or $f_\lambda^\ell(\alpha_2) = \beta$). Therefore, there exists an infinite sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of elements in $C(\alpha_1, \alpha_2; \beta) \subset \overline{K}$ for which the corresponding exponents $m_k, n_k \in \mathbb{N}$ satisfying

$$f_{\lambda_k}^{m_k}(\alpha_1) = f_{\lambda_k}^{n_k}(\alpha_2) = \beta, \tag{6.2.2}$$

must also satisfy

$$\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty. \tag{6.2.3}$$

Now, applying Proposition 5.3 to the two relations in (6.2.2) yields the following inequalities:

$$\widehat{h}_{\lambda_k}(\alpha_1) \le \frac{2h(\alpha_1) + 2h(\beta)}{d^{m_k}} \text{ and } \widehat{h}_{\lambda_k}(\alpha_2) \le \frac{2h(\alpha_2) + 2h(\beta)}{d^{n_k}}. \tag{6.2.4}$$

Inequalities (6.2.4) combined with equation (6.2.3) lead to the desired conclusion. □

6.3. **Equality for all local heights.** First, for each $v \in \Omega_K$, we fix an extension of $|\cdot|_v$ to an absolute value on the entire $\overline{K}$; also, we define the local canonical heights $\widehat{h}_{\lambda,v}$ as in (4.4.1). Next, we state [Ghi, Theorem 4.1], which is instrumental in our proof.

Theorem 6.3. *With the above notation for K, α_1, α_2, assume there exist infinitely many $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \widehat{h}_{\lambda_k}(\alpha_1) = \lim_{k\to\infty} \widehat{h}_{\lambda_k}(\alpha_2) = 0$. Then for each $\lambda \in \overline{K}$ and for each $v \in \Omega_K$, we have that*

$$\widehat{h}_{\lambda,v}(\alpha_1) = \widehat{h}_{\lambda,v}(\alpha_2). \tag{6.3.1}$$

Proposition 6.2 yields the existence of an infinite sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that the hypotheses in Theorem 6.3 are met. Therefore, we conclude that equation (6.3.1) holds for each absolute value $|\cdot|_v$ and for each $\lambda \in \overline{K}$.

6.4. **Finishing the proof of Theorem 2.1 in the special case when the degree is a power of the characteristic.** In this Subsection, we work under the additional hypothesis that $d = p^\ell$ for some $\ell \in \mathbb{N}$; also, we know that equation (6.3.1) holds. With this new assumption, we will prove that

- either $\alpha_1 = \alpha_2$, i.e., conclusion (i) from Theorem 2.1 holds.
- or $\alpha_1 - \beta, \alpha_2 - \beta \in \overline{\mathbb{F}}_p$, i.e., conclusion (ii) from Theorem 2.1 holds.

Since $d = p^\ell$, the iterates of $f_\lambda(z) = z^d + \ell$ have the following explicit form:

$$f_\lambda^n(z) = z^{p^{\ell n}} + \sum_{i=0}^{n-1} \lambda^{p^{i\ell}}, \tag{6.4.1}$$

for each $n \in \mathbb{N}$. The formula (6.4.1) will help in the proof of the following result.

Proposition 6.4. *With the above assumptions, we have $\alpha_1 - \alpha_2 \in \overline{\mathbb{F}}_p$.*

Proof. We argue by contradiction. Assume $\alpha_1 - \alpha_2 \notin \overline{\mathbb{F}}_p$, which means there exists some $v \in \Omega_K$ such that $|\alpha_1 - \alpha_2|_v > 1$. Since $|\alpha_1 - \alpha_2|_v \leq \max\{|\alpha_1|_v, |\alpha_2|_v\}$, we may assume without loss of generality that $|\alpha_2|_v > 1$. We choose $n \in \mathbb{N}$ large enough so that

$$|\alpha_1 - \alpha_2|_v^{d^n} > |\alpha_2|_v. \tag{6.4.2}$$

We let $\lambda \in \overline{K}$ such that $f_\lambda^n(\alpha_2) = 0$. By equation (6.4.1), we have:

$$\alpha_2^{p^{n\ell}} + \lambda^{p^{\ell(n-1)}} + \lambda^{p^{\ell(n-2)}} + \cdots + \lambda^{p^\ell} + \lambda = 0. \tag{6.4.3}$$

First, we compute the local canonical height of α_2. Since $|\alpha_2|_v > 1$, equation (6.4.3) implies:

$$|\lambda|_v = |\alpha_2|_v^{\frac{1}{p^\ell}} = |\alpha_2|_v^{\frac{1}{d}}. \tag{6.4.4}$$

As $|f_\lambda^n(\alpha_2)|_v = 0 < |\lambda|_v^{1/d}$, we apply Lemma 5.1 (iii) and use (6.4.4) to find:

$$\widehat{h}_{\lambda,v}\left(f_\lambda^n(\alpha_2)\right) = \frac{\log|\lambda|_v}{d} = \frac{\log|\alpha_2|_v}{d^2}. \tag{6.4.5}$$

Combining equations (6.4.5) and (4.4.2), we obtain:

$$\widehat{h}_{\lambda,v}(\alpha_2) = \frac{\log|\alpha_2|_v}{d^{n+2}}. \tag{6.4.6}$$

Next, we compute the local canonical height of α_1. Using equations (6.4.1) and (6.4.3), we express $f_\lambda^n(\alpha_1)$ as follows:

$$f_\lambda^n(\alpha_1) = \alpha_1^{p^{n\ell}} + \sum_{i=0}^{n-1} \lambda^{p^{i\ell}} = \alpha_1^{p^{n\ell}} - \alpha_2^{p^{n\ell}} = (\alpha_1 - \alpha_2)^{d^n}. \tag{6.4.7}$$

Using equations (6.4.7), (6.4.2) and (6.4.4), we deduce that

$$|f_\lambda^n(\alpha_1)|_v = |\alpha_1 - \alpha_2|_v^{d^n} > |\alpha_2|_v > |\alpha_2|_v^{\frac{1}{d}} = |\lambda|_v. \tag{6.4.8}$$

Equation (6.4.8) allows us to apply Lemma 5.1 (ii), which yields

$$\widehat{h}_{\lambda,v}\left(f_\lambda^n(\alpha_1)\right) = \log|f_\lambda^n(\alpha_1)|_v = d^n \log|\alpha_1 - \alpha_2|_v. \tag{6.4.9}$$

From equations (6.4.9) and (4.4.2), it follows that

$$\widehat{h}_{\lambda,v}(\alpha_1) = \log|\alpha_1 - \alpha_2|_v. \tag{6.4.10}$$

Finally, comparing the local heights from equations (6.4.6) and (6.4.10), our initial choice of n in inequality (6.4.2) shows that $\widehat{h}_{\lambda,v}(\alpha_1) > \widehat{h}_{\lambda,v}(\alpha_2)$. This contradicts equation (6.3.1). Therefore, there is no $v \in \Omega_K$ such that $|\alpha_1 - \alpha_2|_v > 1$. Hence, $\alpha_1 - \alpha_2 \in \overline{\mathbb{F}}_p$, as desired. □

We recall the existence of the infinite sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\overline{K}$ for which there exist corresponding exponents $m_k, n_k \in \mathbb{N}$ such that

$$f_{\lambda_k}^{m_k}(\alpha_1) = f_{\lambda_k}^{n_k}(\alpha_2) = \beta \tag{6.4.11}$$

for each $k \in \mathbb{N}$. The next lemma shows that if $m_k = n_k$ for some $k \in \mathbb{N}$, then conclusion (i) from Theorem 2.1 must hold.

Lemma 6.5. *If $m_k = n_k$ for some $k \in \mathbb{N}$, then $\alpha_1 = \alpha_2$.*

Proof. Since $d = p^\ell$, we have that $f_\lambda(z)$ is a permutation polynomial on $\overline{K}$ (for each $\lambda \in \overline{K}$); it follows that f_λ^m also induces a permutation on $\overline{K}$ for each $m \in \mathbb{N}$. Therefore, if $m_k = n_k$, the condition $f_{\lambda_k}^{m_k}(\alpha_1) = f_{\lambda_k}^{n_k}(\alpha_2)$ becomes $f_{\lambda_k}^{m_k}(\alpha_1) = f_{\lambda_k}^{m_k}(\alpha_2)$, which implies $\alpha_1 = \alpha_2$. □

Lemma 6.5 thus shows that alternative (i) from the conclusion of Theorem 2.1 holds if $m_k = n_k$ for some $k \in \mathbb{N}$. From now on, we assume that $m_k \neq n_k$ for all $k \in \mathbb{N}$.

The following result is an easy application of the formula for $f_\lambda^n(z)$ from equation (6.4.1). Furthermore, Lemma 6.6 will also be used in Section 8 in the proof of Theorem 2.2.

Lemma 6.6. *Let $\alpha, \lambda \in \overline{K}$ and let $\gamma \in \overline{\mathbb{F}}_p$. If for some $m \in \mathbb{N}$ the following holds:*

$$f_\lambda^m(\alpha) = \alpha + \gamma, \tag{6.4.12}$$

then α and $\alpha + \gamma$ are periodic under the action of f_λ.

Proof. Equation (6.4.1) and our hypothesis (6.4.12) together yield:

$$f_\lambda^{2m}(\alpha) = f_\lambda^m(\alpha+\gamma) = \alpha^{p^{m\ell}} + \gamma^{p^{m\ell}} + \sum_{i=0}^{m-1} \lambda^{p^{i\ell}} = \gamma^{p^{m\ell}} + f_\lambda^m(\alpha) = \gamma^{p^{m\ell}} + \gamma + \alpha. \tag{6.4.13}$$

An easy induction (iterating the computation from (6.4.13) and employing equation (6.4.12)) shows that for each $n \geq 1$,

$$f_\lambda^{mn}(\alpha+\gamma) = \sum_{j=0}^{n-1} \gamma^{p^{jm\ell}} + \alpha. \tag{6.4.14}$$

We choose $r \in \mathbb{N}$ such that $\gamma \in \mathbb{F}_{p^r}$. Equation (6.4.14) then implies that for each $n \geq 1$,

$$f_\lambda^{mn}(\alpha+\gamma) - \alpha \in \mathbb{F}_{p^r}. \tag{6.4.15}$$

Equation (6.4.15) shows that the sequence $\{f_\lambda^{mn}(\alpha+\gamma)-\alpha\}_{n\geq 1}$ takes only finitely many values. Therefore, there exist positive integers $\ell_1 < \ell_2$ such that $f_\lambda^{m\ell_1}(\alpha+\gamma) = f_\lambda^{m\ell_2}(\alpha+\gamma)$. Hence, $\alpha+\gamma$ is preperiodic under the action of f_λ. By equation (6.4.12), α must also be preperiodic under the action of f_λ. Finally, because f_λ induces a permutation on $\overline{L}$ (as explained in the proof of Lemma 6.5), any preperiodic point is necessarily periodic. Thus, both α and $\alpha+\gamma$ are periodic points for f_λ, as desired. □

The following result is an immediate consequence of Lemma 6.6.

Lemma 6.7. *With the notation as in equation (6.4.11), for each $k \in \mathbb{N}$, the points α_1, α_2, and β are periodic under the action of f_{λ_k}.*

Proof. Let $k \in \mathbb{N}$. By our assumption that $m_k \neq n_k$, we may assume without loss of generality that $m_k < n_k$. Because $f_\lambda(z)$ induces a permutation on $\overline{K}$, the relation (6.4.11) implies

$$\alpha_1 = f_{\lambda_k}^{n_k - m_k}(\alpha_2). \tag{6.4.16}$$

Since $n_k - m_k \geq 1$ and $\alpha_1 - \alpha_2 \in \overline{\mathbb{F}}_p$ by Proposition 6.4, equation (6.4.16) provides the hypothesis needed to apply Lemma 6.6. The lemma then implies that α_1 and α_2 are periodic under the action of f_{λ_k}. From (6.4.11), β is also periodic under the action of f_{λ_k}. □

We already proved that $\alpha_1 - \alpha_2 \in \overline{\mathbb{F}}_p$; the following result shows that $\alpha_1 - \beta \in \overline{\mathbb{F}}_p$ as well, which gives the complete picture for alternative (ii) from the conclusion of Theorem 2.1.

Proposition 6.8. *We must have $\alpha_1 - \beta \in \overline{\mathbb{F}}_p$.*

Proof. Fix $k \in \mathbb{N}$, and let $\lambda := \lambda_k$. According to Lemma 6.7, α_1 and β are periodic under the action of f_λ. Using formula (6.4.1), we compute:

$$f_\lambda^n(\alpha_1) - f_\lambda^n(\beta) = (\alpha_1 - \beta)^{p^{n\ell}} \text{ for each } n \in \mathbb{N}. \tag{6.4.17}$$

Since both sequences $\{f_\lambda^n(\alpha_1)\}_{n\geq 1}$ and $\{f_\lambda^n(\beta)\}_{n\geq 1}$ have finitely many elements, the left-hand side of (6.4.17) takes only finitely many values as $n \in \mathbb{N}$ varies. Thus, the right-hand side of (6.4.17) also takes finitely many values, that is, $(\alpha_1 - \beta)^{p^{n_1\ell}} = (\alpha_1 - \beta)^{p^{n_2\ell}}$ for some positive integers $n_1 < n_2$. Consequently, we obtain $\alpha_1 - \beta \in \overline{\mathbb{F}}_p$. □

To summarize, Propositions 6.4 and 6.8 yield that when $d = p^\ell$, the hypotheses in Theorem 2.1 imply that at least one of the alternatives (i)-(iii) from its conclusion must hold. Therefore, for the remainder of the proof, we assume d is not a power of p.

6.5. **Conclusion for our proof of Theorem 2.1.** We have now reduced the proof to the case where $\alpha_1 \notin \overline{\mathbb{F}}_p$ and d is not a power of the characteristic p. The following result, which is [Ghi, Proposition 5.1], provides the final step.

Theorem 6.9. *With the above notation for K, α_1, α_2, assume equality (6.3.1) holds for each absolute value $|\cdot|_v$ and for each $\lambda \in \overline{K}$. If d is not a power of $p = \mathrm{char}(K)$, and at least one of α_1, α_2 is not in $\overline{\mathbb{F}}_p$, then $\alpha_1^d = \alpha_2^d$.*

This completes the proof of Theorem 2.1.

7. Proof of Theorem 2.4

In this Section, we prove Theorem 2.4. We work under its hypotheses: $d \geq 2$ is an integer, L is a field of characteristic p, and $\alpha, \beta \in L$. As before, we consider the family of polynomials $f_\lambda(z) = z^d + \lambda$ parameterized by $\lambda \in \overline{L}$. We will prove that the set

$$C(\alpha;\beta) = \left\{\lambda \in \overline{L} : \text{ there exists } m \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha) = \beta\right\} \tag{7.0.1}$$

is infinite.

7.1. **Proof of Theorem 2.4 when the degree is a power of the characteristic of our field.** We first prove Theorem 2.4 under the additional assumption that $d = p^\ell$ for some $\ell \in \mathbb{N}$.

Lemma 7.1. *With the above assumptions, $C(\alpha;\beta)$ is an infinite set.*

Proof. Since $d = p^\ell$, the equation $f_\lambda^m(\alpha) = \beta$ has the following explicit form due to (6.4.1):

$$\alpha^{p^{m\ell}} + \sum_{i=0}^{m-1} \lambda^{p^{i\ell}} = \beta. \tag{7.1.1}$$

As the equation (7.1.1) is separable (in λ), it has distinct roots for each m. Therefore, the set $C(\alpha;\beta)$, being the union of these roots over all $m \in \mathbb{N}$, is infinite. □

7.2. **Conclusion of our proof for Theorem 2.4.** In light of Lemma 7.1, it suffices to prove Theorem 2.4 under the assumption that d is *not* a power of p.

We argue by contradiction and assume $C(\alpha;\beta)$ is finite. Let $C(\alpha;\beta) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ for some $r \in \mathbb{N}$. For each $m \in \mathbb{N}$, the expansion of $f_\lambda^m(\alpha)$ from Lemma 3.2 yields the equation:

$$P_{m,\alpha}(\lambda) = \lambda^{d^{m-1}} + \sum_{i=1}^{d^{m-1}-1} c_{m,i}(\alpha) \cdot \lambda^{d^{m-1}-i} + \alpha^{d^m} = \beta. \tag{7.2.1}$$

Since $C(\alpha;\beta) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, for each $m \in \mathbb{N}$, there exist some nonnegative integers $e_{m,1}, \ldots, e_{m,r}$ such that the polynomial $P_{m,\alpha}(u) - \beta \in L[u]$ factors as follows in $\overline{L}[u]$:

$$P_{m,\alpha}(u) - \beta = \prod_{i=1}^{r} (u - \lambda_i)^{e_{m,i}}. \tag{7.2.2}$$

Note that the left-hand side (as a polynomial in $L[u]$) is a monic polynomial by (7.2.1), which justifies the right-hand side of (7.2.2) because all the roots of $P_{m,\alpha}(u) - \beta$ are among $\lambda_1, \ldots, \lambda_r$. On the other hand, we have the recurrence formula:

$$P_{m+1,\alpha}(u) = P_{m,\alpha}(u)^d + u,$$

which combined with equation (7.2.2) yields

$$\left(\prod_{i=1}^{r} (u - \lambda_i)^{e_{m,i}} + \beta \right)^d + u - \beta = \prod_{i=1}^{r} (u - \lambda_i)^{e_{m+1,i}}. \tag{7.2.3}$$

To interpret the equation (7.2.3), we consider the subgroup Γ of $\mathbb{G}_m^2(L(u))$ spanned by all the elements $(u - \lambda_i, 1)$ and $(1, u - \lambda_i)$ for $i = 1, \ldots, r$. We also consider the curve V inside $\mathbb{G}_m^2$ defined by the equation

$$(x + \beta)^d = y + (\beta - u). \tag{7.2.4}$$

Since the equation (7.2.4) for V is linear in y, the curve V is geometrically irreducible. Also, equation (7.2.3) shows that $V(L(u)) \cap \Gamma$ is infinite, as it contains all points of the form

$$(P_{m,\alpha}(u) - \beta, P_{m+1,\alpha}(u) - \beta) = \left(\prod_{i=1}^{r} (u - \lambda_i)^{e_{m,i}}, \prod_{i=1}^{r} (u - \lambda_i)^{e_{m+1,i}} \right). \tag{7.2.5}$$

Note that $\beta^d \neq \beta - u$ because $\beta \in L$ and u is a transcendental variable over $\overline{L}$; thus, equation (7.2.4) shows that the curve V is not the translate of an algebraic subgroup of $\mathbb{G}_m^2$.

Indeed, the equation of any translate of a proper subtorus of $\mathbb{G}_m^2$ (defined over $\overline{L(u)}$) is of the form $x^a y^b = \zeta$ for some $\zeta \in \overline{L(u)}^*$ and some integers a and b, not both equal to 0.

We now employ [MS04, Theorem B], which states that the intersection $V \cap \Gamma$ is a finite union of sets of the form

$$A(\eta_0, \eta_1, \dots, \eta_s; k_1, \dots, k_s) \cdot H, \tag{7.2.6}$$

where $H \subseteq \Gamma$ is a subgroup, $s \in \mathbb{N}$, while for some given $\eta_0, \eta_1, \dots, \eta_s \in \mathbb{G}_m^2(\overline{L(u)})$ and some $k_1, \dots, k_s \in \mathbb{N}$, we have that

$$A(\eta_0, \eta_1, \dots, \eta_s, k_1, \dots, k_s) := \left\{ \eta_0 \cdot \prod_{i=1}^{s} \eta_i^{p^{k_i n_i}} : n_i \in \mathbb{N} \right\}. \tag{7.2.7}$$

In formula (7.2.6), we use the notation $C_1 \cdot C_2$ for any two subsets $C_1, C_2 \subset \mathbb{G}_m^2$ to denote the set $\{c_1 \cdot c_2 : c_i \in C_i \text{ for } i = 1, 2\}$.

Furthermore, as explained in [MS04, Remark 2.11], there exists some $N \in \mathbb{N}$ such that

$$\eta_i^N \in \Gamma \text{ for } i = 0, 1, \dots, s. \tag{7.2.8}$$

Since V is an irreducible curve which is not a translate of an algebraic subgroup of $\mathbb{G}_m^2$, the subgroups H from equation (7.2.6) must be finite (see also [Ghi19, Corollary 2.3]). Therefore, at the expense of replacing each set (7.2.6) by finitely many other sets of the form (7.2.6), we may assume from now on that H is the trivial subgroup of Γ.

As V is a curve, we have $s = 1$ in equations (7.2.6) and (7.2.7) by [Ghi19, Corollary 2.3]. Thus, the intersection $V \cap \Gamma$ is a union of finitely many sets of the form

$$A(\eta_0, \eta_1; k) = \left\{ \eta_0 \cdot \eta_1^{p^{kn}} : n \in \mathbb{N} \right\},$$

for some given η_0, η_1 satisfying (7.2.8) and some $k \in \mathbb{N}$. Next, we write each $\eta_j := (\gamma_{j,1}, \gamma_{j,2})$ for $j = 0, 1$. Using (7.2.8), we can express

$$\gamma_{j,1}^N =: \prod_{i=1}^{r} (u - \lambda_i)^{a_{j,i}}, \tag{7.2.9}$$

for some integers $a_{j,i}$ for $j = 0, 1$ and $i = 1, \dots, r$. Therefore, the N-th power of the first components (in $\mathbb{G}_m^2$) of the points in $A(\eta_0, \eta_1; k)$ are of the form

$$\prod_{i=1}^{r} (u - \lambda_i)^{a_{0,i} + a_{1,i} p^{kn}}, \tag{7.2.10}$$

as we vary n in $\mathbb{N}$. In particular, the degrees in u of the polynomials appearing in equation (7.2.10) form the set

$$\left\{ A_0 + A_1 p^{kn} : n \in \mathbb{N} \right\}, \tag{7.2.11}$$

where $A_0 := \sum_{i=1}^{r} a_{0,i}$ and $A_1 := \sum_{i=1}^{r} a_{1,i}$. On the other hand, we already know that $V \cap \Gamma$ contains the elements from equation (7.2.5); in particular, each $P_{m,\alpha}(u) - \beta$ (as we vary $m \in \mathbb{N}$) appears as the first component of an element from the intersection $V \cap \Gamma$. We have (see (7.2.1)) that the degree (in u) of $P_{m,\alpha}(u) - \beta$ is d^{m-1}. Therefore, the set

$$\left\{ d^{m-1} : m \in \mathbb{N} \right\}$$

is contained in the union of finitely many sets of the form (7.2.11). In particular, for some choice of A_0 and A_1 (and $k \in \mathbb{N}$), there exist infinitely many $m \in \mathbb{N}$ such that the equation

$$A_0 + A_1 p^{kn} = d^{m-1}, \tag{7.2.12}$$

has some solution $n \in \mathbb{N}$. However, because d is *not* a power of p, equation (7.2.12) can have only finitely many solutions. This finiteness is a consequence of deep results in Diophantine analysis; for instance, it is a very special case of Laurent's famous theorem on the Mordell-Lang conjecture for algebraic tori *in characteristic* 0 [Lau84, Théorème 2]. Alternatively, the same conclusion follows from the theory of the S-unit equation [Sch90, Theorem 1.1]. This final contradiction shows that the assumption that the equations (7.2.1) (as we vary $m \in \mathbb{N}$) have only finitely many roots $\lambda \in \overline{L}$ is untenable. Therefore, the set $C(\alpha; \beta)$ must be infinite.

This concludes our proof of Theorem 2.4.

8. Proof of Theorem 2.2

In this Section, we prove Theorem 2.2. We work under its stated hypotheses: L is a field of characteristic p with points $\alpha_1, \alpha_2, \beta \in L$ satisfying $\alpha_1 \neq \alpha_2$, and $d = p^\ell$ for some $\ell \in \mathbb{N}$. As before, we let $f_\lambda(z) = z^d + \lambda$ for each $\lambda \in \overline{L}$ and consider the set:

$$C(\alpha_1, \alpha_2; \beta) = \left\{ \lambda \in \overline{L} \colon \text{ there exist } m, n \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta \right\}.$$

We let $\delta_1 := \alpha_2 - \alpha_1$ and $\delta_2 := \beta - \alpha_1$. Furthermore, we assume

$$\delta_1 \in \mathbb{F}_q^* \text{ and } \delta_2 \in \mathbb{F}_q \tag{8.0.1}$$

for some finite subfield $\mathbb{F}_q \subseteq L$. Our goal is to prove that $C(\alpha_1, \alpha_2; \beta)$ is infinite if the system of two equations:

$$\begin{cases} \delta_1 &= \sum_{i=0}^{s_1-1} \gamma^{p^{ik\ell}} \\ \delta_2 &= \sum_{i=0}^{s_2-1} \gamma^{p^{ik\ell}} \end{cases} \tag{8.0.2}$$

has a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Moreover, we will also show that if the system (8.0.2) has no such solution, then $C(\alpha_1, \alpha_2; \beta)$ is *empty*. We split our proof of Theorem 2.2 over several Subsections of Section 8.

8.1. **Strategy for proving Theorem 2.2.**

We obtain the desired conclusion from Theorem 2.2 by first finding explicit conditions which are equivalent with the existence of *at least one* $\lambda \in C(\alpha_1, \alpha_2; \beta)$ (see Proposition 8.3); more precisely, the existence of some $\lambda \in C(\alpha_1, \alpha_2; \beta)$ is equivalent to a solution to the system (8.0.2). Then we prove that one solution to the system (8.0.2) leads to infinitely many solutions to the system (8.0.2) and in turn, this leads to infinitely many $\lambda \in C(\alpha_1, \alpha_2; \beta)$ (see Proposition 8.4).

We will also prove that equation (8.0.1) alone *does not always* imply that the set $C(\alpha_1, \alpha_2; \beta)$ is infinite (see Proposition 8.5); in other words, there are examples of δ_1, δ_2 as in (8.0.1) such that the system (8.0.2) has *no solutions* and therefore, $C(\alpha_1, \alpha_2; \beta)$ is *empty*.

8.2. **Reductions in our proof of Theorem 2.2.** Since $\alpha_2-\alpha_1, \beta-\alpha_1 \in \overline{\mathbb{F}}_p$, then $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p(\alpha_1)$; so, without loss of generality (see also Lemma 3.4), we may assume that $L = \overline{\mathbb{F}}_p(\alpha_1)$.

We first obtain more precise information regarding $\delta_1 = \alpha_2 - \alpha_1$ and $\delta_2 = \beta - \alpha_1$ under the assumption that $C(\alpha_1, \alpha_2; \beta)$ is *nonempty*. To that end, let $\lambda \in C(\alpha_1, \alpha_2; \beta)$ and let $m, n \in \mathbb{N}$ such that

$$f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta. \tag{8.2.1}$$

We will see next that we can, in fact, *always* assume $m > n$ in (8.2.1).

Lemma 8.1. *With the notation as in (8.2.1), we can assume that $m > n$.*

Proof. First, we note that by Lemma 6.5, the case $m = n$ implies $\alpha_1 = \alpha_2$, which contradicts the hypothesis in Theorem 2.2. Thus, it suffices to show that if $n > m$ in (8.2.1), then we can replace m by a suitable integer larger than n so that (8.2.1) holds.

Assume $n > m$ in (8.2.1). Since f_λ induces a permutation on $\overline{L}$, equation (8.2.1) implies:

$$f_\lambda^{n-m}(\alpha_2) = \alpha_1. \tag{8.2.2}$$

Since $\alpha_1 - \alpha_2 = -\delta_1 \in \overline{\mathbb{F}}_p$, we can apply Lemma 6.6 to equation (8.2.2) to deduce that both α_1 and α_2 are periodic points for f_λ. Let $t_0 \in \mathbb{N}$ be the period of α_1 under the action of f_λ. Define $m' := m + nt_0$. By periodicity of α_1, we have $f_\lambda^{m'}(\alpha_1) = f_\lambda^m(\alpha_1) = \beta$. This gives us another instance of (8.2.1), namely, $f_\lambda^{m'}(\alpha_1) = f_\lambda^n(\alpha_2) = \beta$. Since $m' > n$, this concludes our proof of Lemma 8.1. □

8.3. **Different points in the orbit which differ by an element from $\overline{\mathbb{F}}_p$.** According to Lemma 8.1, we may assume that $m > n$ in equation (8.2.1).

Because $d = p^\ell$, we recall from (6.4.1) that for every $n \in \mathbb{N}$,

$$f_\lambda^n(z) = z^{p^{n\ell}} + \sum_{i=0}^{n-1} \lambda^{p^{i\ell}}. \tag{8.3.1}$$

In particular, we have (for each z and ϵ)

$$f_\lambda^n(z+\epsilon) = z^{p^{n\ell}} + \epsilon^{p^{n\ell}} + \sum_{i=0}^{n-1} \lambda^{p^{i\ell}} = f_\lambda^n(z) + \epsilon^{p^{n\ell}}. \tag{8.3.2}$$

Since $f_\lambda(z) = z^d + \lambda = z^{p^\ell} + \lambda$ induces a permutation on $\overline{L}$ and $m > n$, equation (8.2.1) yields that $f_\lambda^{m-n}(\alpha_1) = \alpha_2$. We recall that

$$\delta_1 := \alpha_2 - \alpha_1 \in \mathbb{F}_q^* \text{ and } \delta_2 := \beta - \alpha_1 \in \mathbb{F}_q; \text{ also, we let} \tag{8.3.3}$$

$$k \in \mathbb{N} \text{ be minimal with the property that } f_\lambda^k(\alpha_1) - \alpha_1 \in \mathbb{F}_q. \tag{8.3.4}$$

Note that $f_\lambda^m(\alpha_1) = \beta = \alpha_1 + \delta_2$ and $f_\lambda^{m-n}(\alpha_1) = \alpha_2 = \alpha_1 + \delta_1$; so, equation (8.3.3) ensures that k from (8.3.4) is well-defined. We let $\gamma := f_\lambda^k(\alpha_1) - \alpha_1$; due to (8.3.4), we have

$$\gamma \in \mathbb{F}_q. \tag{8.3.5}$$

For each $a \geq 0$, we write $u_a := f_\lambda^{ka}(\alpha_1) - \alpha_1$; clearly, $u_0 = 0$ and $u_1 = \gamma$. A simple induction on a using equation (8.3.2) establishes the recurrence relation

$$u_{a+1} = u_a^{p^{k\ell}} + \gamma. \tag{8.3.6}$$

To show the inductive step, we compute $u_{a+1} = f_\lambda^{k(a+1)}(\alpha_1) - \alpha_1$ and so,

$$u_{a+1} = f_\lambda^k\left(f_\lambda^{ka}(\alpha_1)\right) - \alpha_1 = f_\lambda^k(\alpha_1 + u_a) - \alpha_1 \overset{(8.3.2)}{=} f_\lambda^k(\alpha_1) + u_a^{p^{k\ell}} - \alpha_1.$$

Since $f_\lambda^k(\alpha_1) - \alpha_1 = \gamma$, we obtain the recurrence relation (8.3.6). In particular, equation (8.3.6) shows that

$$f_\lambda^{ka}(\alpha_1) - \alpha_1 = \sum_{i=0}^{a-1} \gamma^{p^{ik\ell}} \in \mathbb{F}_q. \tag{8.3.7}$$

Lemma 8.2. *Let $s \in \mathbb{N}$ such that $f_\lambda^s(\alpha_1) - \alpha_1 \in \mathbb{F}_q$. Then $k \mid s$.*

Proof. If $k \nmid s$, then there exists $a \geq 0$ and $r \in \{1, \ldots, k-1\}$ such that $s = ka + r$. Then equation (8.3.7) yields that $u_a = f_\lambda^{ka}(\alpha_1) - \alpha_1 \in \mathbb{F}_q$. Using equation (8.3.2), we compute:

$$f_\lambda^s(\alpha_1) = f_\lambda^{ka+r}(\alpha_1) = f_\lambda^r\left(f_\lambda^{ka}(\alpha_1)\right) = f_\lambda^r(\alpha_1 + u_a) = f_\lambda^r(\alpha_1) + u_a^{p^{r\ell}}. \tag{8.3.8}$$

Since $u_a \in \mathbb{F}_q$ and $f_\lambda^s(\alpha_1) - \alpha_1 \in \mathbb{F}_q$ (our hypothesis), we deduce $f_\lambda^r(\alpha_1) - \alpha_1 \in \mathbb{F}_q$ from equation (8.3.8). However, since $1 \leq r \leq k-1$, this contradicts the minimality of k from (8.3.4). Therefore, we must have that $k \mid s$, as desired. □

Using Lemma 8.2 along with equation (8.3.3), we conclude that $m - n = ks_1$ and $m = ks_2$ for some positive integers $s_1 < s_2$.

8.4. **The defining system of two equations and one unknown.** According to equations (8.3.3) and (8.3.7), the original condition from (8.2.1) translates to *two* equations:

$$\delta_1 = f_\lambda^{m-n}(\alpha_1) - \alpha_1 = f_\lambda^{ks_1}(\alpha_1) - \alpha_1 = u_{ks_1} \text{ and so,}$$

$$\delta_1 = \sum_{i=0}^{s_1-1} \gamma^{p^{ik\ell}}; \text{ and} \tag{8.4.1}$$

$$\delta_2 = f_\lambda^m(\alpha_1) - \alpha_1 = f_\lambda^{ks_2}(\alpha_1) - \alpha_1 = u_{ks_2} \text{ and so,}$$

$$\delta_2 = \sum_{i=0}^{s_2-1} \gamma^{p^{ik\ell}}. \tag{8.4.2}$$

We summarize our findings so far in the following Proposition.

Proposition 8.3. *With the notation as in Theorem 2.2, assume $\delta_1 = \alpha_2 - \alpha_1 \neq 0$ and $\delta_2 = \beta - \alpha_1$ are contained in $\mathbb{F}_q$. Then the set $C(\alpha_1, \alpha_2; \beta)$ is nonempty if and only if the system (8.0.2) has a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.*

Proof. First, assume $C(\alpha_1, \alpha_2; \beta)$ is nonempty and let $\lambda \in C(\alpha_1, \alpha_2; \beta)$. Then $f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta$ for some $m, n \in \mathbb{N}$. Since $\alpha_1 \neq \alpha_2$, Lemma 6.5 implies $m \neq n$, and by Lemma 8.1, we may assume $m > n$. This leads to the system (8.0.2) (see also equations (8.4.1) and (8.4.2)) to have a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Furthermore, the assumption $\delta_1 \neq 0$ ensures that $\gamma \neq 0$ due to equation (8.4.1).

Now, for the converse implication, we assume the system (8.0.2) has a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. From equation (8.3.1) we solve for $\lambda \in \overline{L}$ such that

$$f_\lambda^k(\alpha_1) = \alpha_1 + \gamma. \tag{8.4.3}$$

Using again (8.3.1) coupled with (8.4.3), along with equations (8.4.1) and (8.4.2), we obtain

$$f_\lambda^{ks_1}(\alpha_1) = \alpha_1 + \delta_1 = \alpha_2 \text{ and} \tag{8.4.4}$$

$$f_\lambda^{ks_2}(\alpha_1) = \alpha_1 + \delta_2 = \beta. \tag{8.4.5}$$

So, choosing $m := ks_2$ and $n := k(s_2 - s_1)$, we get that equation (8.2.1) holds; therefore, $\lambda \in C(\alpha_1, \alpha_2; \beta)$, as desired.

This concludes our proof of Proposition 8.3. □

8.5. **One solution to our system generates infinitely many parameters.** We continue with the notation as in Subsection 8.4. The next Proposition shows that once there exists *one* solution $\gamma \in \mathbb{F}_q^*$ (and $k, s_1, s_2 \in \mathbb{F}_q$) to the system of equations (8.4.1) and (8.4.2), then $C(\alpha_1, \alpha_2; \beta)$ is infinite.

Proposition 8.4. *If there exists a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_q^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to the equations (8.4.1) and (8.4.2), then $C(\alpha_1, \alpha_2; \beta)$ is infinite.*

Proof. We write $q := p^r$ for some $r \in \mathbb{N}$. We note that once we have a solution (γ, k, s_1, s_2) to the system (8.4.1)-(8.4.2), $(\gamma, k + r, s_1, s_2)$ also solves the above system. However, this new solution to the system leads to a different element in $C(\alpha_1, \alpha_2; \beta)$. Indeed, for any such solution (γ, k, s_1, s_2), the corresponding parameter $\lambda \in C(\alpha_1, \alpha_2; \beta)$ satisfies

$$f_\lambda^k(\alpha_1) = \alpha_1 + \gamma,\ f_\lambda^{ks_1}(\alpha_1) = f_\lambda^{m-n}(\alpha_1) = \alpha_1 + \delta_1 \text{ and } f_\lambda^{ks_2}(\alpha_1) = f_\lambda^m(\alpha_1) = \alpha_1 + \delta_2. \tag{8.5.1}$$

Combined with (8.3.1), the first equation from (8.5.1) yields that

$$\alpha_1^{p^{k\ell}} + \sum_{i=0}^{k-1} \lambda^{p^{i\ell}} = \alpha_1 + \gamma. \tag{8.5.2}$$

Equation (8.5.2) in λ is a separable equation of degree $p^{(k-1)\ell}$ and hence has distinct solutions. So, increasing k leads to additional solutions to the new equation (8.5.2). Therefore, we have infinitely many elements in $C(\alpha_1, \alpha_2; \beta)$ simply assuming the existence of one solution (γ, k, s_1, s_2) to the system of equations (8.4.1) and (8.4.2). □

8.6. **Conclusion of our proof for Theorem 2.2.** If the system (8.0.2) has no solutions, then $C(\alpha_1, \alpha_2; \beta)$ must be empty by Proposition 8.3. Now, if the system (8.0.2) has a solution, then Proposition 8.4 yields that there are actually infinitely many $\lambda \in C(\alpha_1, \alpha_2; \beta)$.

This concludes our proof of Theorem 2.2.

8.7. **Sometimes the set $C(\alpha_1, \alpha_2; \beta)$ is empty.** Next, we show that condition (8.0.1) alone from Theorem 2.2 does *not* always guarantee the existence of infinitely many $\lambda \in C(\alpha_1, \alpha_2; \beta)$; in other words, there are instances when the system (8.0.2) is not solvable (despite the fact that $\alpha_2 - \alpha_1, \beta - \alpha_1 \in \overline{\mathbb{F}}_p$) and therefore, $C(\alpha_1, \alpha_2; \beta)$ is empty.

Proposition 8.5. *Let $d = p^\ell$ and let $\alpha_1, \alpha_2, \beta \in L$. If $\delta_1 = \alpha_2 - \alpha_1$ and $\delta_2 = \beta - \alpha_1$ simultaneously satisfy the following conditions:*

(1) $\delta_1, \delta_2 \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$,
(2) $\delta_1 - \delta_2 \notin \mathbb{F}_p$, and
(3) $\frac{\delta_1}{\delta_2} \notin \mathbb{F}_p$,

then $C(\alpha_1, \alpha_2; \beta)$ is empty.

Remark 8.6. For each prime $p > 2$, one can construct examples where Proposition 8.5 applies. For instance, choosing $(\delta_1, \delta_2) = (\epsilon, 2\epsilon + 1)$ for any $\epsilon \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ satisfies the conditions (1)-(3) in Proposition 8.5. Therefore, the corresponding set $C(\alpha_1, \alpha_2; \beta)$ is empty, even though the points $\alpha_1, \alpha_2, \beta$ satisfy the alternative (ii) from the conclusion of Theorem 2.1.

Proof of Proposition 8.5. We argue by contradiction and assume $C(\alpha_1, \alpha_2; \beta) \neq \emptyset$. Therefore, there exists $\lambda \in \overline{L}$ and $m, n \in \mathbb{N}$ such that $f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta$. As shown in Lemma 8.1, we may assume that $m > n$. By Proposition 8.3, this triple $(\lambda, m, n) \in \overline{L} \times \mathbb{N} \times \mathbb{N}$ leads to a solution $(\gamma, k, s_1, s_2) \in \mathbb{F}_{p^2} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to the system of equations (8.4.1) and (8.4.2). Note that, due to (8.3.5), we can assume $\gamma \in \mathbb{F}_{p^2}$ because $\delta_1, \delta_2 \in \mathbb{F}_{p^2}$. We will show that there are no solutions $(\gamma, k, s_1, s_2) \in \mathbb{F}_{p^2} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to the system (8.4.1)-(8.4.2). The proof is divided into two cases based on the parity of $k \cdot \ell$.

Lemma 8.7. *With the above notation, there are no solutions to the system (8.4.1)-(8.4.2) if $k \cdot \ell$ is even.*

Proof of Lemma 8.7. Since any sought solution γ of the system (8.4.1)-(8.4.2) lives in $\mathbb{F}_{p^2}$, we have $\gamma^{p^{ik\ell}} = \gamma$ for each $i \geq 0$. So, the system (8.4.1)-(8.4.2) simplifies to

$$s_1 \cdot \gamma = \delta_1 \text{ and } s_2 \cdot \gamma = \delta_2. \tag{8.7.1}$$

However, (8.7.1) implies $\delta_1/\delta_2 = s_1/s_2$, which contradicts condition (3) from the hypotheses of Proposition 8.5. Thus, no solutions exist when $k\ell$ is even. □

Lemma 8.8. *With the above notation, there are no solutions to the system (8.4.1)-(8.4.2) if $k \cdot \ell$ is odd.*

Proof of Lemma 8.8. In this case, we know that for any even $s \in \mathbb{N}$, we have that

$$\sum_{i=0}^{s-1} \gamma^{p^{ik\ell}} = \frac{s}{2} \cdot \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\gamma), \tag{8.7.2}$$

while for any odd $s \in \mathbb{N}$, we have that

$$\sum_{i=0}^{s-1} \gamma^{p^{ik\ell}} = \gamma + \frac{s-1}{2} \cdot \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\gamma). \tag{8.7.3}$$

So, if s_j is even for some $j = 1, 2$, then (8.7.2) shows that the system (8.4.1)-(8.4.2) leads to $\delta_j \in \mathbb{F}_p$, which contradicts condition (1) from the hypotheses of Proposition 8.5. If both s_1 and s_2 are odd, then (8.7.2) yields that $\delta_1 - \delta_2 \in \mathbb{F}_p$, which contradicts condition (2) from the hypotheses of Proposition 8.5. In all cases, we reach a contradiction, so no solution to the system (8.4.1)-(8.4.2) can exist if $k\ell$ is odd. □

Combining Lemmas 8.7 and 8.8 concludes our proof of Proposition 8.5. □

9. The case when all the points live in a finite field

Throughout this Section, we work with $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$ and the family of polynomials $f_\lambda(z) := z^d + \lambda$ parameterized by $\lambda \in \overline{\mathbb{F}}_p$. Our goal is to determine whether the set

$$C(\alpha_1, \alpha_2; \beta) = \left\{\lambda \in \overline{\mathbb{F}}_p : \text{ there exist } m, n \in \mathbb{N} \text{ such that } f_\lambda^m(\alpha_1) = f_\lambda^n(\alpha_2) = \beta\right\}. \tag{9.0.1}$$

is infinite. Note that restricting λ to $\overline{\mathbb{F}}_p$ is sufficient, as Lemma 3.4 implies that if $f_\lambda^m(\alpha_1) = \beta$, then we have that $\lambda \in \overline{\mathbb{F}_p(\alpha_1, \beta)} = \overline{\mathbb{F}}_p$ because $\alpha_1, \beta \in \overline{\mathbb{F}}_p$.

Since Theorem 2.2 provides an explicit answer when $d = p^\ell$ (everything depends on the existence of a solution to a system (1.3.2) of equations), we assume that d is *not* a power of p. Under this hypothesis, as long as $\alpha_1^d \neq \alpha_2^d$, there is no visible dynamical relation *globally* between α_1 and α_2 with respect to the *entire* family of polynomials $f_\lambda(z)$. So, drawing on the intuition from Theorem 1.1, one would expect that if $\alpha_1^d \neq \alpha_2^d$, then $C(\alpha_1, \alpha_2; \beta)$ is finite. However, based on extensive computation (of more than 100 examples), we believe the *opposite* is true.

Conjecture 9.1. *Let $d \geq 2$ be an integer which is not a power of p and let $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$. For any $\lambda \in \overline{\mathbb{F}}_p$, let $f_\lambda(z) := z^d + \lambda$. Then the set $C(\alpha_1, \alpha_2; \beta)$ (see (9.0.1)) is infinite.*

This Section is organized as follows. In Subsection 9.1, we describe the algorithm used for testing various cases in Conjecture 9.1. In Subsection 9.2, we formulate Conjecture 9.2, which is a refinement of Conjecture 9.1 in the special case $\alpha_1 = \beta$. We gathered in Subsection 9.3 some of the many examples we tested, all of which support both Conjectures 9.1 and 9.2. In Subsection 9.4, we present an example for a different family of polynomials, which in turn suggests Question 9.9. Finally, in Subsection 9.5, we conclude our paper with a brief discussion about the collision of multiple orbits and formulate Question 9.10.

9.1. Algorithm for testing Conjecture 9.1.

Since we are now studying the case where $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$, we fix a sufficiently large q such that $\alpha_1, \alpha_2, \beta \in \mathbb{F}_q$. For a fixed α and $n \in \mathbb{N}$, recall that $P_{n,\alpha}(\lambda) := f_\lambda^n(\alpha)$ is a polynomial in λ with degree d^{n-1}. By definition, $f_\lambda^m(\alpha_1) = \beta$ if and only if λ is a root of $P_{m,\alpha_1}(\lambda) - \beta = 0$. Directly finding the roots of the polynomials $P_{m,\alpha_1}(\lambda) - \beta$ and $P_{n,\alpha_2}(\lambda) - \beta$ is often infeasible due to their high degrees. We now describe a more efficient algorithm for finding values of λ in $C(\alpha_1, \alpha_2; \beta)$ by searching for common factors of these polynomials over finite fields.

The algorithm proceeds by checking, for each irreducible polynomial $g(\lambda)$ over a finite field $\mathbb{F}_q$, whether it divides $P_{n,\alpha}(\lambda) - \beta$ for some $n \in \mathbb{N}$. This check is performed efficiently in the quotient ring $\mathbb{F}_q[\lambda]/\langle g(\lambda)\rangle$. For a given $\alpha, \beta \in \mathbb{F}_q$ and an irreducible polynomial $g(\lambda) \in \mathbb{F}_q[\lambda]$, the procedure is as follows:

(a) We recursively compute the sequence of remainders $r_n := f_\lambda^n(\alpha) \pmod{g(\lambda)}$ by setting $r_0 = \alpha$ and computing $r_n = r_{n-1}^d + \lambda$ in $\mathbb{F}_q[\lambda]/\langle g(\lambda)\rangle$ for $n \geq 1$.
(b) If $r_n = \beta$ for some $n \in \mathbb{N}$, then $g(\lambda)$ divides $P_{n,\alpha}(\lambda) - \beta$ and the algorithm stops for this particular polynomial $g(\lambda)$. Consequently, every root of $g(\lambda) = 0$ is a solution to $f_\lambda^n(\alpha) = \beta$.
(c) If we encounter a remainder r_n that has appeared previously (i.e., $r_n = r_i$ for some $i < n$) and *none* of the remainders $r_1, \ldots, r_n$ are equal to β, a cycle has been detected. We conclude that β will not be reached, and we terminate the search for this polynomial

$g(\lambda)$. Also, as an aside, the pair (i, n) (where i and n are minimal with the property that $f_\lambda^i(\alpha) = f_\lambda^n(\alpha)$) is called the *preperiodicity portrait* for α (under the action of f_λ).

The length of the sequence before repetition is bounded by $q^{\deg(g)}$. For practical implementation, we must set a maximum number of iterations. If this threshold is reached without finding β or a cycle, the algorithm terminates inconclusively for $g(\lambda)$.

To find elements in $C(\alpha_1, \alpha_2, \beta)$, we apply this procedure to both α_1 and α_2. For each monic irreducible polynomial $g(\lambda) \in \mathbb{F}_q[\lambda]$, we search for an integer m such that $g(\lambda)$ divides $P_{m,\alpha_1}(\lambda) - \beta$. If successful, we then search for an integer n such that $g(\lambda)$ also divides $P_{n,\alpha_2}(\lambda) - \beta$. If both searches succeed, all roots of $g(\lambda)$ belong to the set $C(\alpha_1, \alpha_2; \beta)$.

9.2. **The case where $\alpha_1 = \beta$.** We describe a special case in which permutation polynomials naturally arise. Suppose $\alpha_1 = \beta \in \mathbb{F}_q$. We are searching for $\lambda \in \overline{\mathbb{F}_p}$ such that for some $m, n \in \mathbb{N}$, the following equalities hold:

$$f_\lambda^m(\alpha_1) = \alpha_1 \quad \text{and} \quad f_\lambda^n(\alpha_2) = \alpha_1.$$

We now explain how first condition, $f_\lambda^m(\alpha_1) = \alpha_1$, is automatically satisfied for certain choices of λ. Suppose the polynomial $P_{n,\alpha_2}(\lambda) - \alpha_1$ has an irreducible factor $h(\lambda)$ of degree k over $\mathbb{F}_q$. Let $\lambda_0 \in \mathbb{F}_{q^k}$ be any root of $h(\lambda)$. If $\gcd(d, q^k - 1) = 1$, the map $z \mapsto z^d$ is a permutation of $\mathbb{F}_{q^k}$, which implies that $f_{\lambda_0}(z) = z^d + \lambda_0$ is also a permutation of $\mathbb{F}_{q^k}$. Since $\alpha_1 \in \mathbb{F}_q \subseteq \mathbb{F}_{q^k}$, then α_1 must be periodic under f_λ; thus, $f_{\lambda_0}^m(\alpha_1) = \alpha_1$ for some $m \in \mathbb{N}$. The argument shows that when $\gcd(d, q^k - 1) = 1$, each root $\lambda_0 \in \mathbb{F}_{q^k}$ of $h(\lambda)$ satisfies $f_{\lambda_0}^m(\alpha_1) = f_{\lambda_0}^n(\alpha_2) = \alpha_1$ for some $m, n \in \mathbb{N}$. Therefore, to prove that the set $C(\alpha_1, \alpha_2; \alpha_1)$ is infinite, it suffices to show that the polynomials $P_{n,\alpha_2}(\lambda) - \alpha_1$ for $n = 1, 2, 3, \ldots$ have irreducible factors of arbitrarily large degrees k satisfying $\gcd(d, q^k - 1) = 1$. We state this prediction as a conjecture.

Conjecture 9.2. *Suppose $\alpha, \beta \in \mathbb{F}_q$. For $\lambda \in \overline{\mathbb{F}}_p$, let $f_\lambda(z) := z^d + \lambda$. Define, as before, $P_{n,\alpha}(\lambda) = f_\lambda^n(\alpha)$; so, $P_{n,\alpha}(\lambda) \in \mathbb{F}_q[\lambda]$ is a polynomial of degree d^{n-1}. Suppose $\gcd(d, q-1) = 1$. Then for each $M > 0$, there is some $n \in \mathbb{N}$ such that the polynomial $P_{n,\alpha}(\lambda) - \beta$ has an irreducible factor $g(\lambda) \in \mathbb{F}_q[\lambda]$ of degree $k > M$ satisfying $\gcd(d, q^k - 1) = 1$.*

The hypothesis $\gcd(d, q-1) = 1$ is necessary. Indeed, if $\gcd(d, q-1) > 1$, then $\gcd(d, q^k - 1) > 1$ for each $k \geq 1$. If $\gcd(d, q-1) = 1$, then we can find infinitely many integers $k \geq 1$ such that $\gcd(d, q^k - 1) = 1$, so the conclusion of Conjecture 9.2 makes sense. Note that Conjecture 9.2 implies Conjecture 9.1 in the special case when $\alpha_1 = \beta$.

9.3. **Computational evidence.** We list several examples that provide evidence for both Conjecture 9.1 and Conjecture 9.2. When counting irreducible polynomials over a finite field with specific properties, we always restrict our attention to monic irreducible polynomials.

Example 9.3. We work over $\mathbb{F}_2$. Let $d = 3$, $\alpha_1 = 1$, $\alpha_2 = 0$, and $\beta = 1$. We are in the special case when $\alpha_1 = \beta$. Since $\gcd(3, 2^k - 1) = 1$ if and only if $k \in \mathbb{N}$ is odd, we seek any odd-degree irreducible factors of the polynomial $P_{n,0}(\lambda) - 1 = f_\lambda^n(0) - 1$. Any root of such a factor will belong to $C(1, 0; 1)$. Table 1 lists the degrees of the irreducible factors of $P_{n,0}(\lambda) - 1$ for $1 \leq n \leq 11$; newly appearing odd degrees are highlighted.

The presence of factors with large odd degrees, such as 16189 and 42859 for $n = 11$, strongly suggests that $C(1, 0; 1)$ is infinite. Thus, Table 1 numerically supports Conjecture 9.2.

TABLE 1. Degrees of irreducible factors of $P_{n,0}(\lambda) - 1$ for $n = 1, \ldots, 11$

n	Degrees of irreducible factors (all factors are simple)
1	**1**
2	**3**
3	1, 3, **5**
4	**27**
5	1, 38, 42
6	18, **21**, **43**, **71**, 90
7	1, 5, 38, **121**, 564
8	3, **97**, 214, **375**, 1498
9	1, 12, 16, **1205**, **5327**
10	3, **15**, 22, 22, 34, **61**, 82, **161**, 240, 334, 428, **4429**, 13852
11	1, **16189**, **42859**

As direct factorization of $P_{n,0}(\lambda)-1$ (a polynomial of degree 3^{n-1}) is computationally intensive, we also use the general algorithm. Table 2 shows the number of irreducible polynomials $g(\lambda)$ of a given degree that divide $\gcd(P_{m,1}(\lambda) - 1, P_{n,0}(\lambda) - 1)$ for some $m, n \in \mathbb{N}$.

TABLE 2. Irreducible polynomials that divide $\gcd(P_{m,1}(\lambda) - 1, P_{n,0}(\lambda) - 1)$

degree of the irreducible polynomial	1	2	3	4	5	6	7	8	9	10	11	12	13
number of successful polynomials	1	0	2	0	3	0	7	0	31	1	89	4	325

The final entry shows 325 distinct irreducible polynomials of degree 13 over $\mathbb{F}_2$. These alone yield $325 \cdot 13 = 4225$ different values of $\lambda \in \overline{\mathbb{F}}_2$ in the set $C(1, 0; 1)$. Even without Table 1, we see that Table 2 supports Conjecture 9.1.

Example 9.4. We work over $\mathbb{F}_5$. Let $d = 3$, $\alpha_1 = 2$, $\alpha_2 = 1$, and $\beta = 2$. Again, we are in the special case $\alpha_1 = \beta$. Next, $\gcd(3, 5^k - 1) = 1$ if and only if $k \in \mathbb{N}$ is odd. We seek any odd-degree irreducible factors of the polynomial $P_{n,1}(\lambda) - 2 = f_\lambda^n(1) - 2$. Table 3 lists the degrees of irreducible factors of $P_{n,1}(\lambda) - 2$ for each $1 \le n \le 12$; newly appearing odd degrees are highlighted. The presence of factors with large odd degrees, such as 163341 for $n = 12$, strongly suggests that $C(2, 1; 2)$ is infinite. Thus, Table 3 numerically supports Conjecture 9.2.

Example 9.5. We work over $\mathbb{F}_3$. Let $d = 2$, $\alpha_1 = 0$, $\alpha_2 = 1$, and $\beta = 2$. Table 4 shows the number of irreducible polynomials of a given degree dividing $\gcd(P_{m,0}(\lambda) - 2, P_{n,1}(\lambda) - 2)$ for some $m, n \in \mathbb{N}$. The steadily growing counts suggest that $C(0, 1; 2)$ is infinite, thus supporting Conjecture 9.1.

Example 9.6. We work over $\mathbb{F}_9$ and let $\epsilon \in \mathbb{F}_9 \setminus \mathbb{F}_3$ with $\epsilon^2 = -1$. Let $d = 2$, $\alpha_1 = 1$, $\alpha_2 = \epsilon + 1$, and $\beta = \epsilon$. The growing counts in Table 5 suggest that $C(1, \epsilon + 1; \epsilon)$ is infinite, again supporting Conjecture 9.2.

Example 9.7. We work over $\mathbb{F}_8$ and let $\xi \in \mathbb{F}_8 \setminus \mathbb{F}_2$ be an element that satisfies $\xi^3 = \xi + 1$. Let $d = 3$, $\alpha_1 = 1$, $\alpha_2 = \xi$, and $\beta = \xi^2 + \xi + 1$. Table 6 shows the count of irreducible polynomials of a given degree over $\mathbb{F}_8$ that divide $\gcd(P_{m,1}(\lambda) - (\xi^2 + \xi + 1), P_{n,\xi}(\lambda) - (\xi^2 + \xi + 1))$. The growing numbers suggest that $C(1, \xi; \xi^2 + \xi + 1)$ is infinite, which supports Conjecture 9.1.

TABLE 3. Degrees of irreducible factors of $P_{n,1}(\lambda) - 2$ for $n = 1, \dots, 12$

n	Degrees of irreducible factors (all factors are simple)
1	**1**
2	1, 2
3	**3**, 6
4	4, **7**, 16
5	1, 1, **11**, 22, **45**
6	2, **5**, 10, 226
7	26, 52, 250, **401**
8	1, 3, 3, 4, 5, 6, 13, 23, 64, **95**, **149**, **353**, 1468
9	1, 20, **27**, **757**, 1082, 4674
10	2, 4, **21**, 1632, 3932, 14092
11	1, 6, **99**, 106, **205**, 280, 446, 778, 2370, 22642, 32116
12	3, 3, 4, 7, 38, **71**, 13680, **163341**

TABLE 4. Irreducible polynomials that divide $\gcd(P_{m,0}(\lambda) - 2, P_{n,1}(\lambda) - 2)$

degree of the irreducible polynomial	1	2	3	4	5	6	7	8	9	10	11	12	13
number of successful polynomials	2	1	1	4	1	4	7	15	24	29	53	70	120

TABLE 5. Irreducible polynomials that divide $\gcd(P_{m,1}(\lambda) - \epsilon, P_{n,\epsilon+1}(\lambda) - \epsilon)$

degree of the irreducible polynomial	1	2	3	4	5	6
number of successful polynomials	3	1	4	7	40	60

This example also exhibits an interesting feature: there appear to be considerably more values of λ whose minimal polynomial over $\mathbb{F}_8$ has an odd degree compared to an even degree. While we do not have a full explanation for this phenomenon, this parity imbalance also highlights the difficulty of Conjecture 9.1. We note that for each $\lambda \in \mathbb{F}_{8^{2k+1}}$ (for each $k \in \mathbb{N}$), the polynomial f_λ induces a permutation on $\mathbb{F}_{8^{2k+1}}$ and therefore, the problem of colliding orbits becomes a question of having *one* periodic cycle (see also Subsection 9.2) in $\mathbb{F}_{8^{2k+1}}$ (for f_λ) containing *all three* points $\alpha_1, \alpha_2, \beta$. The numerical evidence from this example suggests that it is *more likely* for the three points to belong to the same periodic cycle, rather than for there to be different preperiodicity portraits for the orbits of α_1 and α_2, both of which contain β.

TABLE 6. Irreducible polynomials that divide $\gcd(P_{m,1}(\lambda) - (\xi^2 + \xi + 1), P_{n,\xi}(\lambda) - (\xi^2 + \xi + 1))$

degree of the irreducible polynomial	1	2	3	4	5	6
number of successful polynomials	4	2	63	7	2265	31

Example 9.8. We work over $\mathbb{F}_5$. Let $d = 10$, $\alpha_1 = 1$, $\alpha_2 = 2$, and $\beta = 3$. We consider this case because $p \mid d$, which one may suspect has a different answer; after all, the case $d = p^\ell$ *does* exhibit a special behavior (see Theorem 2.2). Table 7 shows the count of irreducible

polynomials of a given degree over $\mathbb{F}_5$ that divide $\gcd(P_{m,1}(\lambda)-3, P_{n,2}(\lambda)-3)$. Once again, the growing numbers suggest that $C(1,2;3)$ is infinite, thus providing support for Conjecture 9.1.

TABLE 7. Irreducible polynomials that divide $\gcd(P_{m,1}(\lambda)-3, P_{n,2}(\lambda)-3)$

degree of the irreducible polynomial	1	2	3	4	5	6	7	8	9
number of successful polynomials	1	1	1	2	8	13	18	43	103

9.4. **Numerical evidence for a more general question.** We switch now to a different family of polynomials:

$$g_\lambda(z) = z^3 + z + \lambda \text{ (parameterized by } \lambda \in \overline{\mathbb{F}_5}); \tag{9.4.1}$$

also, we consider two starting points α_1, α_2 and one target point β. As before, we are interested in whether the set

$$C_g(\alpha_1, \alpha_2; \beta) := \left\{\lambda \in \overline{\mathbb{F}_5} : \text{ there exist } m, n \in \mathbb{N} \text{ such that } g_\lambda^m(\alpha_1) = g_\lambda^n(\alpha_2) = \beta\right\} \tag{9.4.2}$$

is infinite. For $\alpha_1 = 1$, $\alpha_2 = 3$ and $\beta = 2$, we define the corresponding recurrence polynomials (for all $m, n \in \mathbb{N}$):

$$P_{g,m,1}(\lambda) := g_\lambda^m(1) \text{ and } P_{g,n,3}(\lambda) := g_\lambda^n(3).$$

Table 8 shows the count of (monic) irreducible polynomials of a given degree over $\mathbb{F}_5$ that divides $\gcd\left(P_{g,m,1}(\lambda) - 2, P_{g,n,3}(\lambda) - 2\right)$. The steadily growing numbers suggest that the set $C_g(1,3;2) \subseteq \overline{\mathbb{F}}_5$ corresponding to this polynomial $f_\lambda(z) = z^3 + z + \lambda$ is infinite.

TABLE 8. Irreducible polynomials that divide $\gcd(P_{g,m,1}(\lambda) - 2, P_{g,n,3}(\lambda) - 2)$

degree of the irreducible polynomial	1	2	3	4	5	6	7	8	9
number of successful polynomials	1	0	1	5	6	17	24	32	114

This leads us to believe that the following Question has a positive answer.

Question 9.9. Let $g \in \overline{\mathbb{F}}_p[z]$ be a polynomial of degree $d \geq 2$, which is not an additive polynomial. We consider the family of polynomials $g_\lambda(z) = g(z) + \lambda$, parameterized by $\lambda \in \overline{\mathbb{F}}_p$. Is it true that for each $\alpha_1, \alpha_2, \beta \in \overline{\mathbb{F}}_p$, the set

$$C_g(\alpha_1, \alpha_2; \beta) := \left\{\lambda \in \overline{\mathbb{F}}_p : \text{ there exist } m, n \in \mathbb{N} \text{ such that } g_\lambda^m(\alpha_1) = g_\lambda^n(\alpha_2) = \beta\right\} \tag{9.4.3}$$

is infinite?

A positive answer to Question 9.9 supports the relevance of condition (4) from Conjecture 2.6.

9.5. **Collision of multiple orbits.** We conclude the paper with a new question that leads us to an uncharted territory in the study of collision of orbits. Conjecture 9.1 considers the intersection of two orbits, which is the focus of this paper. This naturally leads to a more general question. So, given a polynomial $g \in \overline{\mathbb{F}}_p[z]$ of degree d, which is not an additive polynomial, we let $g_\lambda(z) := g(z) + \lambda$ be a family of polynomials parameterized by $\lambda \in \overline{\mathbb{F}}_p$. Then for any integer $s \geq 2$ and any given $\alpha_1, \ldots, \alpha_s, \beta \in \overline{\mathbb{F}}_p$, we define:

$$C_g(\alpha_1, \ldots, \alpha_s; \beta) := \{\lambda \in \overline{\mathbb{F}}_p : \text{ there exists } n_i \in \mathbb{N} \text{ such that } f_\lambda^{n_i}(\alpha_i) = \beta \text{ for } i = 1, \ldots, s\}. \tag{9.5.1}$$

Question 9.10. Given a polynomial $g \in \overline{\mathbb{F}}_p[z]$ of degree $d \geq 2$, which is not an additive polynomial, then using the notation from (9.5.1), what is the smallest integer $s \geq 2$ with the property that there exist $\alpha_1, \ldots, \alpha_s, \beta \in \overline{\mathbb{F}}_p$ such that the corresponding set $C_g(\alpha_1, \ldots, \alpha_s; \beta)$ is finite?

We present one of the many examples we tested for our Question 9.10.

Example 9.11. We work over $\mathbb{F}_5$, and let $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\beta = 4$. We consider the two families of polynomials $f_\lambda(z) = z^3 + \lambda$ and $g_\lambda(z) = z^4 + z + \lambda$. Tables 9 and 10 show the counts of (monic) irreducible polynomials in both settings. The growing counts suggest that the sets $C_f(1, 2, 3; 4)$ and $C_g(1, 2, 3; 4)$ are both infinite. We also see a qualitative difference: the counts for $C_f(1, 2, 3; 4)$ exhibit a parity imbalance (with a clear bias for odd degrees), while the counts for $C_g(1, 2, 3; 4)$ show a general upward trend as the degree increases.

We believe that the imbalance from Table 9 for odd degree polynomials may again be a consequence of the fact that for each $\lambda \in \mathbb{F}_{5^{2k+1}}$, the polynomial $f_\lambda(z)$ induces a permutation polynomial on $\mathbb{F}_{5^{2k+1}}$. It seems far more likely that $\alpha_1, \alpha_2, \alpha_3, \beta$ all live in the same periodic cycle (under the action of $f_\lambda(z)$) rather than any other configuration of the preperiodicity portraits for the orbits of $\alpha_1, \alpha_2, \alpha_3$ intersecting at β. The latter scenario could happen when $\lambda \in \mathbb{F}_{5^{2k}}$ since the points may no longer be periodic under the action of $f_\lambda(z)$.

TABLE 9. Irreducible polynomials that divide $\gcd(P_{f,m,1}(\lambda) - 4, P_{f,n,2}(\lambda) - 4, P_{f,k,3}(\lambda) - 4)$

degree of the irreducible polynomial	1	2	3	4	5	6	7	8
number of successful polynomials	0	1	8	2	154	6	2732	28

TABLE 10. Irreducible polynomials that divide $\gcd(P_{g,m,1}(\lambda) - 4, P_{g,n,2}(\lambda) - 4, P_{g,k,3}(\lambda) - 4)$

degree of the irreducible polynomial	1	2	3	4	5	6	7	8
number of successful polynomials	1	0	3	5	7	10	24	43

Our numerical data suggest that the original intuition for collision of orbits must be revised when working over $\overline{\mathbb{F}}_p$ due to the underlying finite combinatorics. More precisely, when the starting points α_j (for $j = 1, \ldots, s$), the target point β, and the parameter λ all belong to a finite field $\mathbb{F}_{p^k}$, we are asking whether β is contained in *each* of the finite subsets $\mathcal{O}_{f_\lambda}(\alpha_j)$ for $1 \leq j \leq s$. Even though this is *unlikely* for any *given* λ, the positive answer becomes *likely* once we look over *all* $\lambda \in \mathbb{F}_{p^k}$. In contrast, when the field of definition for the starting points α_j and for the target point β has positive transcendence degree, the collision of orbits is *unlikely* without a well-defined global dynamical relation between the points (see Theorem 1.1). This new perspective in characteristic p is reminiscent of the likely, unlikely, and impossible intersections studied in [CGMM13], where the transcendence degree of the base field was also the crucial factor in determining the nature of an intersection.

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