CHROMATIC QUASISYMMETRIC FUNCTIONS OF THE PATH GRAPH

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ABSTRACT. We show that the chromatic quasisymmetric function (CQF) of a labeled path graph on n vertices is *not* symmetric unless the labeling is the natural labeling $1, 2, ..., n$ or its reverse $n, ..., 2, 1$. We also show that the star graph $K_{1,n-1}$ with $n \geq 3$ has a nonsymmetric CQF for all labelings.

1. INTRODUCTION

R. Stanley [Sta95] defined the chromatic symmetric function of a graph as an infinite power series generalization of the chromatic polynomial: if a graph G has vertex set $V = \{v_1, \ldots, v_n\}$, with $\mathcal{PC}(G)$ denoting its proper colorings using colors from $\mathbb{P} = \{1, 2, \ldots\}$, then its *chromatic symmetric function* is

$$X(G; \mathbf{x}) = \sum_{c \in \mathcal{PC}(G)} \mathbf{x}^c,$$

where $\mathbf{x}^c = x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_n)}$ with $c(v_i)$ the color of vertex v_i.

The chromatic quasisymmetric function of a graph is a generalization of the chromatic symmetric function. It was introduced by Shareshian and Wachs [SW16] as a tool to solve the Stanley-Stembridge Conjecture [SS93] given in terms of the chromatic symmetric function in [Sta95], which is a central area of research in algebraic combinatorics. Special cases of this conjecture have been solved, for example [CH19, GS01, HP19, HNY20]. More recently, Hikita has a preprint [Hik24] proving the Stanley-Stembridge conjecture in full generality. For the chromatic quasisymmetric function, we have a graph $G = (V, E)$, with $|V| = n$, whose vertices are labeled with $[n] = \{1, 2, ..., n\}$. More precisely, we have a bijective map $L \colon V \to [n]$ that assigns a *label* to each $v \in V$. The data (G, L) constitute a *labeled graph*. Instead of differentiating between the vertices and the labels, we will often identify the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ with $[n] = \{1, 2, \ldots, n\}$.

Given a coloring $c \in \mathcal{PC}(G)$, the *ascent set* of c is:

$$\mathrm{Asc}_G(c) = \{ij \in E \mid i < j \text{ and } c(i) < c(j)\}.$$

The *ascent number* of c is $\mathrm{asc}_G(c) = \# \mathrm{Asc}_G(c)$. Similarly, the *descent set* of c is

$$\mathrm{Des}_G(c) = \{ij \in E \mid i < j \text{ and } c(i) > c(j)\}.$$

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The *descent number* of c is $\mathrm{des}_G(c) = \#\mathrm{Des}_G(c)$. If the graph G is clear from the context, we omit the subscript G. As above, we denote by $\mathbf{x}^c$ the monomial $x_{c(1)}x_{c(2)}\cdots x_{c(n)}$ where $c(i)$ is the color of the vertex labeled $i \in [n]$.

The *chromatic quasisymmetric function (CQF)* of G is

$$X(G;\mathbf{x},q) = \sum_{c\in\mathcal{PC}(G)} \mathbf{x}^c q^{\mathrm{asc}(c)}.$$

We can express $X(G;\mathbf{x},q)$ as

$$X(G;\mathbf{x},q) = \sum_{k\geq 0} X_k(G;\mathbf{x})q^k,$$

where each $X_k(G;\mathbf{x})$ is a quasisymmetric function of degree n. Moreover, $X(G;\mathbf{x},q)$ is a polynomial in q with degree equal to the number of edges of G. Finally, the chromatic quasisymmetric function refines the chromatic symmetric function as $X(G;\mathbf{x},1) = X(G;\mathbf{x})$. See [SW16, Page 502].

In general, there is no known classification of labeled graphs whose chromatic quasisymmetric functions have coefficients that are symmetric. In this paper, we completely resolve this question for the path graph P_n. We also discuss the star graph $K_{n-1,1}$ and give a sufficient (but not necessary) condition for an arbitrary tree to have a nonsymmetric CQF (see Corollary 5.4).

The *natural labeling* of P_n is 1, 2, ..., n or its reverse $n, \ldots, 2, 1$, since reversing labels on the path graph leads to an isomorphic labeled graph. More precisely, if (P_n, L) is a labeled path, then it has a natural labeling if $L(v_i) = i$ for all i or $L(v_i) = n+1-i$ for all i. For example, the two isomorphic natural labelings for P_5 are:

(1)—(2)—(3)—(4)—(5)   (5)—(4)—(3)—(2)—(1).

The path graph below has a "non-natural" labeling:

(3)—(2)—(5)—(1)—(4).

Shareshian and Wachs [SW16] proved that $X(P_n;\mathbf{x},q)$ is symmetric if P_n is labeled with the natural labeling. In this paper, we prove the converse statement, which leads to the following result:

1.1. **Theorem.** *The chromatic quasisymmetric function of* P_n *is symmetric if and only if* P_n *has the natural labeling.*

The principal strategy is to associate to each labeled path graph P_n a *ribbon diagram* with n boxes, which encodes the *ascent-descent pattern* linked with the labeling of the path. Section 3 provides details about this approach.

Outline. In Section 2, we give essential definitions and provide background. In Section 3, we associate a ribbon tableau $\mathrm{RD}(P_n)$ to a given colored labeled path. We prove our main Theorem 1.1 in Section 4. Finally, in Section 5, we treat the CQF of the star graph $K_{1,n-1}$ and present a general result for a class of trees.

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2. Preliminaries

This section reviews the necessary definitions and background material for chromatic quasisymmetric functions (CQFs). We discuss both symmetric and palindromic CQFs. As a sufficient test for a CQF to be palindromic, we study the properties of the *flip map* on the vertex set, given by $i \mapsto n+1-i$.

Following the development and notation in [Sag20], let $\mathbf{x} = \{x_1, x_2, \dots\}$ be a countably infinite set of commuting variables, and let $\mathbb{C}[[\mathbf{x}]]$ be the algebra of formal power series in $\mathbf{x}$. We say $f(x) \in \mathbb{C}[[\mathbf{x}]]$ is *homogeneous of degree n* if the sum of the exponents of each monomial in $f(x)$ equals n, and $f(x)$ is *symmetric* if the coefficient of the monomial $x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_\ell}^{m_\ell}$ (for $i_1, i_2, \dots, i_\ell$ distinct subscripts) is the same as the coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_\ell^{m_\ell}$ in $f(x)$.

Let

$$\mathrm{Sym}_n = \{f(x) \in \mathbb{C}[[\mathbf{x}]] \mid f(x) \text{ is symmetric and homogeneous of degree } n\}.$$

Then *the algebra of symmetric functions* is

$$\mathrm{Sym} = \bigoplus_{n \geq 0} \mathrm{Sym}_n .$$

Sym is well-studied, and there are several interesting bases indexed by integer partitions. The reader is referred to [Sag20], [Sta23], and [Mac15] for a wealth of information on symmetric functions.

We say that $f(x) \in \mathbb{C}[[\mathbf{x}]]$ is *quasisymmetric* if the coefficient of $x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_\ell}^{m_\ell}$ for $i_1 < i_2 < \cdots < i_\ell$ is the same as the coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_\ell^{m_\ell}$ in $f(x)$. Let

$$\mathrm{QSym}_n = \{f(x) \in \mathbb{C}[[\mathbf{x}]] \mid f(x) \text{ is quasisymmetric and homogeneous of degree } n\}.$$

Then the *algebra of quasisymmetric functions* is

$$\mathrm{QSym} = \bigoplus_{n \geq 0} \mathrm{QSym}_n .$$

Note that every symmetric function is also quasisymmetric, but the converse is false. Bases of QSym are indexed by integer compositions.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \vDash n$, the *monomial quasisymmetric function* $M_\alpha \in \mathrm{QSym}_n$ is defined by:

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.$$

Then, QSym_n is a vector space of dimension 2^{n-1}, with basis $\{M_\alpha \mid \alpha \vDash n\}$ for $n \geq 1$ and QSym_0 is spanned by $\{M_0 = 1\}$, where 0 stands for the empty composition.

Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \vDash n$, the *reverse* of α is the composition $\alpha^r = (\alpha_\ell, \alpha_{\ell-1}, \dots, \alpha_1)$. Define an involution ρ on QSym_n by assigning

$$\rho(M_\alpha) = M_{\alpha^r},$$

on the basis elements and extending linearly. The map ρ further extends to an involution of $\mathrm{QSym}_n[q]$.

In this paper, we focus on two special classes of chromatic quasisymmetric functions.

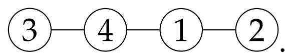
- We say that a CQF $X(G; \mathbf{x}, q)$ is *symmetric* if $X(G; \mathbf{x}, q) \in \mathrm{Sym}[q]$, meaning that the coefficients are symmetric functions (not just quasisymmetric functions).

- We say that a CQF $X(G;\mathbf{x},q)$ is *palindromic* if $X(G;\mathbf{x},q)$ is a palindromic polynomial in q, meaning that the coefficient (living in QSym) of q^i equals the coefficient of q^{m-i} where m is the number of edges in G.

An equivalent definition for symmetry of $X(G;\mathbf{x},q)$ is that the coefficient of M_α in $X(G;\mathbf{x},q)$ is equal to the coefficient of M_β (as polynomials in q) where β is any rearrangement of α.

Shareshian and Wachs [SW16, Proposition 2.6] show that ρ reverses the coefficients of a CQF as a polynomial in q, which we will use in Proposition 2.2. Hence, a CQF Q is palindromic if and only if $\rho(Q) = Q$. It is straightforward to check that a necessary condition for Q to be palindromic is that the coefficient of M_α in Q is equal to the coefficient of M_{α^r} for each composition α.

As an illustration, consider the labeled path graph P_4:



The associated CQF is

$$(5q^3+7q^2+7q+5)M_{(1,1,1,1)} + (2q^3+q^2+q+2)M_{(1,1,2)} +$$
$$(q^3+2q^2+2q+1)M_{(1,2,1)} + (2q^3+q^2+q+2)M_{(2,1,1)} + (q^3+1)M_{(2,2)},$$

which is palindromic, but not symmetric since the coefficients of $M_{(1,1,2)}$ and $M_{(1,2,1)}$ differ.

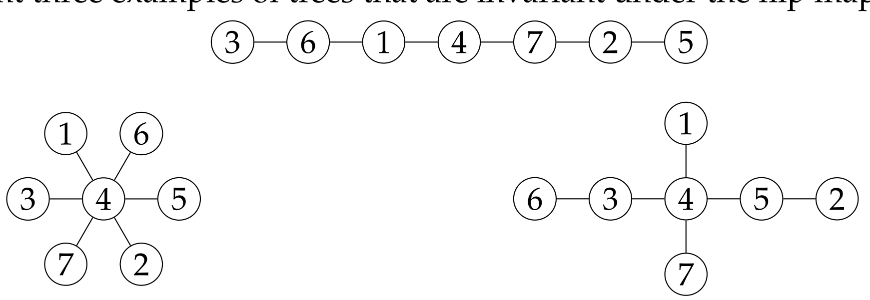
According to [SW16, Corollary 2.8], every symmetric CQF is palindromic. When showing that a particular graph has a nonsymmetric CQF, it is sometimes feasible to show the stronger result that the CQF is nonpalindromic.

On the other hand, showing that a given CQF is palindromic appears to be a difficult task. Below, we present one criterion that guarantees a graph has a palindromic CQF.

Recall that two labeled graphs (G, L) and (G', L') are *isomorphic* if there exists a graph isomorphism $\omega\colon G \to G'$ such that $L'(\omega(v)) = L(v)$ for each vertex $v \in V(G)$. In this case, we write $(G, L) \cong (G', L')$.

2.1. **Definition.** Let (G, L) be a labeled graph with vertex set $V = \{v_1, \ldots, v_n\}$ with the labeling map $L\colon V \to [n]$. The *flip map* f sends (G, L) to the labeled graph (G', L') by defining G' to have the vertex set V and the labeling $L'(v_i) = n + 1 - L(v_i)$. For brevity, we write $f(G) := G'$. We say G is *invariant under the flip map* if $(G, L) \cong (G', L')$.

We present three examples of trees that are invariant under the flip map.



2.2. **Proposition.** *Suppose Q is the CQF of a labeled graph G, and Q' is the CQF of the labeled graph G', obtained from G via the flip map. Then $Q + Q'$ is palindromic.*

Proof. Any proper coloring c of G is also a proper coloring of G' (and vice-versa), corresponding to the same monomial x^c, as the flip map permutes the labels of the vertices of G, but does not change the color of each vertex.

It is straightforward to check that $\mathrm{Asc}_G(c) = \mathrm{Des}_{G'}(c)$. Therefore, we also have that $\mathrm{Asc}_{G'}(c) = \mathrm{Des}_G(c)$, and so

$$Q' = \sum_{c \in \mathcal{PC}(G')} x^c q^{\mathrm{asc}_{G'}(c)} = \sum_{c \in \mathcal{PC}(G)} x^c q^{\mathrm{des}_G(c)} = \rho(Q),$$

where the last equality follows from [Sag20, Theorem 8.5.3 (b)]. Since ρ is an involution, we then have:

$$\rho(Q + Q') = \rho(Q + \rho(Q)) = \rho(Q) + Q = Q' + Q.$$

So, $Q + Q'$ is indeed palindromic. □

The proposition above allows us to conclude the following.

2.3. **Corollary.** *Suppose Q is the CQF of a labeled graph G and suppose Q' is the CQF of the labeled graph $G' = f(G)$. If G is isomorphic to G' as labeled graphs, then Q is palindromic.*

Proof. By Proposition 2.2 and the fact that $Q = Q'$:

$$\rho(Q) = \rho\left(\frac{Q + Q'}{2}\right) = \frac{Q + Q'}{2} = Q,$$

as desired. □

3. Labeled Paths and Ribbon Tableaux

Given a labeled path graph (P_n, L), its vertex set is $V = \{v_1, \ldots, v_n\}$ where $v_i v_{i+1}$ is an edge for $1 \leq i \leq n-1$. We can then define a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ where $\sigma_i := L(v_i)$.

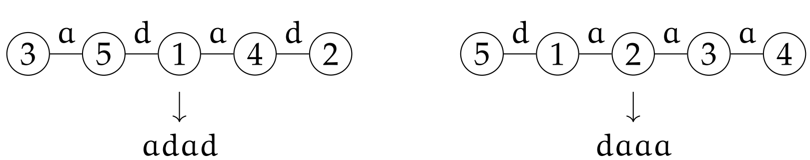
3.1. **Definition.** Given a labeled path P_n with associated permutation $\sigma_1\sigma_2 \cdots \sigma_n$, the *ascent-descent pattern*, or $\mathtt{ad}$-*pattern* for short , is a word $w_1 w_2 \cdots w_{n-1}$, in the alphabet $\{\mathtt{a}, \mathtt{d}\}$, with $w_i = \mathtt{a}$ if $\sigma_i < \sigma_{i+1}$ and $w_i = \mathtt{d}$ if $\sigma_i > \sigma_{i+1}$, for each $1 \leq i \leq n-1$.

For example, consider the two labelings of P_5 below.

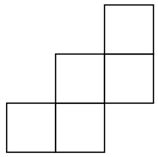
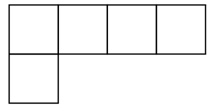
The $\mathtt{ad}$-pattern for the left graph is $\mathtt{adad}$, and for the right graph, $\mathtt{daaa}$.

By definition, the CQF of a path is invariant under permuting the labels but maintaining the order of the labels of adjacent vertices. To understand the CQF of a labeled path, it therefore suffices to focus on its $\mathtt{ad}$-pattern alone.

We can visualize an $\mathtt{ad}$-pattern using a *ribbon diagram* (also called a *rim-hook* or *border strip*) with n boxes as follows. We start with a single box and then sequentially append a new box to the right (resp. above) for each $\mathtt{a}$ (resp. $\mathtt{d}$) in the $\mathtt{ad}$-pattern. For example, the two graphs above have ad-patterns



and hence ribbon diagrams

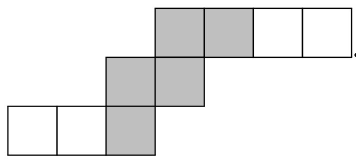
 and 

respectively.

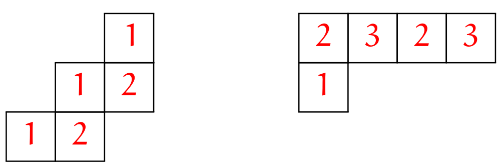
Note that we can completely describe a ribbon diagram with n boxes by a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \vDash n$, where α_i is the number of boxes in row i of the diagram, with row 1 being the bottom row. For example, the two ribbon diagrams above have corresponding compositions of $(2,2,1)$ and $(1,4)$. We call the ribbon diagram whose composition is α the α *ribbon*. For example, the left ribbon diagram above is the $(2,2,1)$ ribbon. We will also refer to the CQF of a labeled path as the CQF of its ribbon diagram.

3.2. **Definition.** Any contiguous collection of boxes of a ribbon diagram D is called a *sub-ribbon* of D, or more specifically, a β sub-ribbon, if it has composition β.

For example, the shaded boxes form a $(1,2,2)$ sub-ribbon of the ribbon diagram:



We can use a ribbon tableau to encode a coloring of a labeled path P_n. Given a proper coloring $c \in \mathcal{PC}(P_n)$ of a labeled path P_n with vertex set $[n]$, we place the color $c(i)$ in the ith box of its ribbon diagram, starting with the bottom, left box, and following the ribbon, creating its *ribbon tableau*. For example, below, we have two colored paths (with the color of each vertex indicated above it) and their corresponding ribbon tableaux.

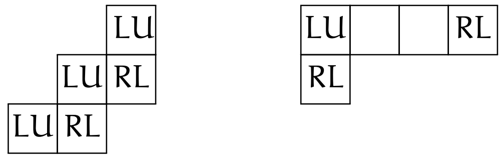


We define $\mathrm{RD}(P_n)$ to be the ribbon diagram associated to a (labeled) path P_n. We define the *proper coloring of* $\mathrm{RD}(P_n)$, denoted $\mathrm{RT}(c)$, to be the ribbon tableau associated to a proper coloring $c \in \mathcal{PC}(P_n)$ with the number in each box called its *color*.

In Section 4, certain boxes of $\mathrm{RD}(P_n)$ will have special significance, so we define them now.

3.3. **Definition.** Given a path P_n, whose vertices are labeled with $[n]$, and corresponding ribbon diagram $\mathrm{RD}(P_n)$, a *left-upper* (LU) *corner* of $\mathrm{RD}(P_n)$ is a box with no box to its left and no box above it. A *right-lower* (RL) *corner* is a box with no box to its right and no box below it.

For example, in the two ribbon diagrams below, the left-upper corners are labeled "LU," and the right-lower corners are labeled "RL."



Observe that the maximum ascent number of a proper coloring c of a graph $G = (V, E)$ equals $|E|$. This maximum is achieved if and only if, for each edge $ij \in E$, we have $c(i) < c(j)$ whenever $i < j$. If $G = P_n$, this happens if and only if the colors of $RT(c)$ *increase left to right in each row* and *increase from top to bottom in each column*. In particular, the color 1 must always be assigned to an LU box.

More generally, if P_n does not have the natural labeling, for a given coloring c and its corresponding colored ribbon tableau $RT(c)$, $asc(c)$ equals the number of adjacent horizontal pairs in $RT(c)$ whose colors increase left to right plus the number of adjacent vertical pairs whose colors increase top to bottom.

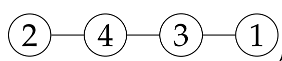
4. THE CHROMATIC QUASISYMMETRIC FUNCTION OF THE PATH GRAPH

For any labeled graph G with n vertices, consider the expansion of $X(G; \mathbf{x}, q)$ in the monomial quasisymmetric function basis:

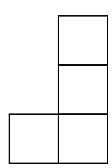
$$X(G; \mathbf{x}, q) = \sum_{\alpha \vDash n} c_\alpha(q) M_\alpha.$$

Note that the coefficients $c_\alpha(q)$ are polynomials in q. If $G = P_n$, then M_α appears in $X(P_n; \mathbf{x}, q)$ if and only if there is a proper coloring of $RD(P_n)$ that uses color i α_i times, for $i = 1, 2, \ldots$. We say such a coloring has *content* α. The coefficient of q^k in $c_\alpha(q)$ equals the number of proper colorings of P_n using content α, with k ascents.

For example, if P_4 is



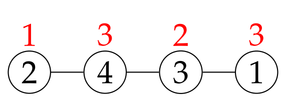
then $RD(P_4)$ is

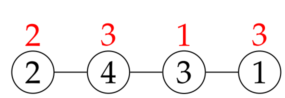


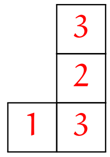
and

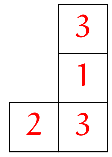
$$\begin{aligned} X(P_4, \mathbf{x}, q) = &(3q^3 + 9q^2 + 9q + 3)M_{(1,1,1,1)} \\ &+ (2q^2 + 3q + 1)M_{(1,1,2)} + (q^3 + 2q^2 + 2q + 1)M_{(1,2,1)} \\ &+ (q^3 + 3q^2 + 2q)M_{(2,1,1)} + (q^2 + q)M_{(2,2)}. \end{aligned}$$

We see that $c_{(1,1,2)}(q) = 2q^2 + 3q + 1$, so there are two colorings of $RD(P_4)$ that yield 2 ascents and use color 1 one time, color 2 one time, and color 3 two times, as shown below.









In what follows, we assume that the vertices of P_n are labeled with $[n]$, but not in the natural order. Hence its ad-pattern is not $aa\cdots a$ or $dd\cdots d$, and so $RD(P_n)$ does not have composition (n) or (1^n).

We prove Theorem 1.1 by analyzing several types of configurations or sub-ribbons that $RD(P_n)$ might contain. First, we can reduce the number of cases with the following:

4.1. **Proposition.** *If a labeled path P_n has a symmetric CQF, then the labeled path obtained by applying the flip map of Definition 2.1 has a symmetric CQF.*

Proof. We extend the definition of the flip map to a colored path graph, P_n. Let $f(P_n)$ be the colored, labeled path obtained from P_n by replacing label i with $n+1-i$, but retaining the coloring. Then

$$X(f(P_n); \mathbf{x}, q) = \sum_{c \in \mathcal{PC}(f(P_n))} \mathbf{x}^c q^{\mathrm{asc}(c)} = \sum_{c \in \mathcal{PC}(P_n)} \mathbf{x}^c q^{\mathrm{des}(c)}.$$

By [SW16, Corollary 2.7], since $X(P_n, \mathbf{x}, q)$ is symmetric, we have that

$$X(P_n, \mathbf{x}, q) = \sum_{c \in \mathcal{PC}(P_n)} \mathbf{x}^c q^{\mathrm{des}(c)}.$$

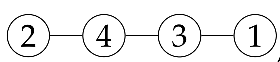
Thus,

$$X(f(P_n); \mathbf{x}, q) = X(P_n; \mathbf{x}, q)$$

and both are symmetric. □

Translating this result to ribbon diagrams, we note that the ad-pattern of $f(P_n)$ is obtained from the ad-pattern of P_n by replacing each 'a' with a 'd' and each 'd' with an 'a'. This corresponds to reflecting the ribbon diagram $RD(P_n)$ across the diagonal that starts at the lower left corner of the ribbon diagram and has slope 1, resulting in the "reflection" of the ribbon diagram.

For example, if P_n is



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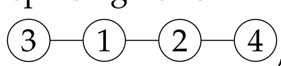
then $f(P_n)$ is obtained from P_n by replacing vertex label i with $4 - i + 1$:



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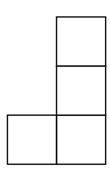
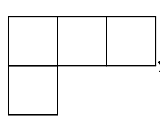
and their corresponding ribbon diagrams are



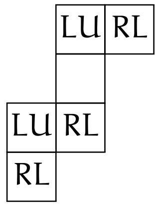
and

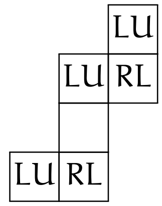

,

respectively, with the second ribbon diagram being the reflection of the first.

Proposition 4.1 allows us to conclude that any statement about the symmetry, or lack thereof, of the CQF of a ribbon diagram immediately applies to the CQF of its reflection.

The proposition below reduces our study to ribbon diagrams with the same number of LU and RL corners. Namely, it shows that the corresponding labeled paths do not have symmetric CQFs if these counts are different. Two examples of such ribbon diagrams are:





4.2. **Proposition.** *Let P_n be a labeled graph such that the number of LU and RL corners of* $\mathrm{RD}(P_n)$ *are different. Then,* $X(P_n; \mathbf{x}, q)$ *is not palindromic, and hence not symmetric.*

Proof. Let k be the number of LU corners and j be the number of RL corners.

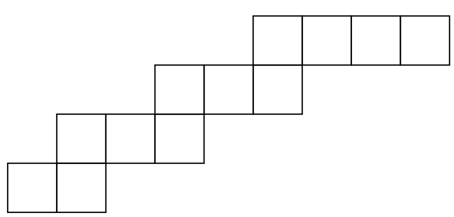
If $k < j$, then $q^{|E|}M_{(k,1^{n-k-j},j)}$ is a term of $X(P_n; \mathbf{x}, q)$. This is because we can construct a coloring c with content $(k, 1^{n-k-j}, j)$ and $\mathrm{asc}(c) = |E|$ as follows. The LU corners can be colored 1, the RL corners can be colored with the largest color, and all other boxes can be colored using the remaining colors so that row and column entries increase. However, $q^{|E|}M_{(j,1^{n-k-j},k)}$ is *not* a term of $X(P_n; \mathbf{x}, q)$ because there is no way to place color 1 in $j > k$ boxes, while maximizing ascents, since, to maximize ascents, color 1 can only be placed in the k LU corners.

If $k > j$, a similar argument shows that $q^{|E|}M_{(k,1^{n-k-j},j)}$ is a term of $X(P_n; \mathbf{x}, q)$, but $q^{|E|}M_{(j,1^{n-k-j},k)}$ is not. Note that the largest color can only be placed in the j RL corners for maximum ascent.

In either case, we have found a composition α such that the coefficients of M_α and M_{α^r} are not equal, and hence $X(P_n; \mathbf{x}, q)$ is not palindromic. □

Proposition 4.2 was proved independently and recast in terms of acyclic directed graphs, sources and sinks in [GPS24, Lemma 4.2]. Note that our LU corners and RL corners correspond to their sources and sinks, respectively.

We say that i consecutive rows (resp. columns) that each contain at least 2 boxes is a *stack* of i rows (resp. columns). If all rows (resp. columns) contain at least two boxes, we say the ribbon diagram consists of *stacked* rows (resp. columns). A ribbon diagram of stacked rows is shown below.



Our next result shows that labeled paths whose ribbon diagrams consist of stacked rows do not have a symmetric CQF.

4.3. **Notation.** If $f(x)$ is a formal power series, we denote the coefficient of x^n in $f(x)$ by $[x^n]f(x)$.

4.4. **Proposition.** *Suppose that P_n is a labeled path whose composition corresponding to* $\mathrm{RD}(P_n)$ *is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_i \geq 2$ for each i and $k \geq 2$. Then,* $X(P_n; \mathbf{x}, q)$ *is not symmetric.*

Proof. Let r, s be the lengths of any two adjacent rows of $RD(P_n)$, i.e. $(r, s) = (\alpha_i, \alpha_{i+1})$. Choose b to be any integer in the interval $[r, n-k-s+2]$. Note this interval is non-empty since $(r-1)+(s-1)+k \leq n$, where the left-hand side counts the number of boxes in rows $i, i+1$, as well as the LU corners in $RD(P_n)$.

To prove that $X(P_n; \mathbf{x}, q)$ is not symmetric, we show that

$$[q^{|E|}M_{(k,1^{n-k})}]X(P_n; \mathbf{x}, q) > [q^{|E|}M_{(1^{b-1},k,1^{n-k-b+1})}]X(P_n; \mathbf{x}, q).$$

Let A denote the set of proper colorings of $RD(P_n)$ with content $(k, 1^{n-k})$ and maximum ascent number $|E|$. Each coloring in A contributes $q^{|E|}M_{(k,1^{n-k})}$ to $X(P_n; \mathbf{x}, q)$. Note that these colorings use color 1 exactly k times and every other color once. The set A is not empty because every row has at least two boxes. Similarly, let B denote the set of proper colorings of $RD(P_n)$ with content $(1^{b-1}, k, 1^{n-k-b+1})$ and maximum ascent number $|E|$. Each coloring in B contributes $q^{|E|}M_{(1^{b-1},k,1^{n-k-b+1})}$ to $X(P_n; \mathbf{x}, q)$. Note that these colorings use color b exactly k times and every other color once.

We extend B to B', the set of (not necessarily proper) colorings of $RD(P_n)$ with content $(1^{b-1}, k, 1^{n-k-b+1})$ such that the rows are strictly increasing left to right, while the columns are weakly increasing top to bottom.

For example, consider the $(3, 3, 4)$ ribbon diagram. Let $i = 1$, so $r = 3$ and $s = 3$. Then, since $k = 3$, we have $b \in [3, 6]$; we choose $b = 4$. With content $(1^3, 3, 1^4)$, the ribbon tableau below, constructed according to the proof of this proposition, is an element of B', but the coloring is not proper, so it is not an element of B.

$$\begin{array}{cccccccc} & & & & 3 & 4 & 5 & 6 \\ & & 4 & 7 & 8 & & & \\ 1 & 2 & 4 & & & & & \end{array}$$

As noted previously, because every coloring $c \in A$ has a maximum ascent number, the colors of c must increase in each row. Since there are k rows and k copies of color 1, the leftmost box of each row must be colored 1.

Define a bijection $\zeta\colon A \to B'$ as follows: Given $c \in A$, identify the unique $k-1$ rows not containing b. Replace the $k-1$ occurrences of 1's in these rows with b's and sort each row so that its colors are increasing left to right. For example, if

$$c_1 = \begin{array}{cccccccc} & & & & 1 & 3 & 4 & 8 \\ & & 1 & 6 & 7 & & & \\ 1 & 2 & 5 & & & & & \end{array}, \qquad \text{then } \zeta(c_1) = \begin{array}{cccccccc} & & & & 1 & 3 & 4 & 8 \\ & & 4 & 6 & 7 & & & \\ 2 & 4 & 5 & & & & & \end{array},$$

which is in B. But if

$$c_2 = \begin{array}{cccccccc} & & & & 1 & 3 & 7 & 8 \\ & & 1 & 5 & 6 & & & \\ 1 & 2 & 4 & & & & & \end{array}, \qquad \text{then } \zeta(c_2) = \begin{array}{cccccccc} & & & & 3 & 4 & 7 & 8 \\ & & 4 & 5 & 6 & & & \\ 1 & 2 & 4 & & & & & \end{array},$$

which is in $B' \setminus B$.

By construction, $\zeta(c) \in B'$ since $\zeta(c)$ could only (potentially) have a descent in a column consisting of the box at the end of a row and the box immediately above it.

$$\begin{array}{ccc} & a_1 & \cdots \\ \cdots & a_2 & \end{array}$$

But every row of $\zeta(c)$ must contain color b exactly once, and the rows are increasing, so $a_1 \leq b$ and $b \leq a_2$. Thus, the columns of $\zeta(c)$ weakly increase from top to bottom.

By definition, each coloring $c' \in B'$ has exactly one b per row, and there will be exactly one row that contains both 1 and b. With this observation, we define the inverse $\zeta^{-1}: B' \to A$ as follows: Given $c' \in B'$, identify the $k-1$ rows not containing 1, replace each occurrence of b in these rows with 1, and sort each row so its colors increase left to right.

Next, we show that there exists a coloring in $B' \setminus B$, from which it follows that

$$[q^{|E|}M_{(k,1^{n-k})}]X(P_n;\mathbf{x},q) = |A| = |B'| > |B| = [q^{|E|}M_{(1^{b-1},k,1^{n-k-b+1})}]X(P_n;\mathbf{x},q).$$

Let $m = n-k+1$ be the maximum color used by colorings in B'. We construct a coloring in $B' \setminus B$ as follows.

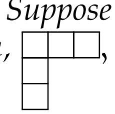
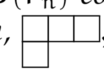
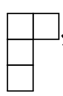
- Fill row i of $RD(P_n)$ with $(1,2,\ldots,r-1,b)$ and note that the row is increasing left to right since $b \geq r$.
- Fill row $i+1$ of $RD(P_n)$ with $(b, m-s+2,\ldots,m-1,m)$ and note that the row is increasing left to right since $b \leq n-k-s+2 = m-s+1$.
- Fill the rest of $RD(P_n)$ arbitrarily such that the coloring is in B'.

Then the coloring is not in B since the column (consisting of two boxes) joining rows i and $i+1$ contains two adjacent copies of color b. □

4.5. **Corollary.** *Suppose that P_n is a labeled path with associated ribbon diagram consisting of at least two stacked columns. Then, $X(P_n;\mathbf{x},q)$ is not symmetric.*

Proof. The result follows by combining Proposition 4.4 and Proposition 4.1. □

The following two propositions show that ribbon diagrams containing certain sub-ribbons correspond to labeled paths that are not symmetric.

4.6. **Proposition.** *Suppose that P_n is a labeled path with vertex set $[n]$. If $RD(P_n)$ contains a $(1,1,3)$ sub-ribbon,* *, begins (on the lower left) with the $(1,3)$ sub-ribbon,* *, or ends with the $(1,1,2)$ sub-ribbon,* *, then $Q = X(P_n;\mathbf{x},q)$ is not symmetric.*

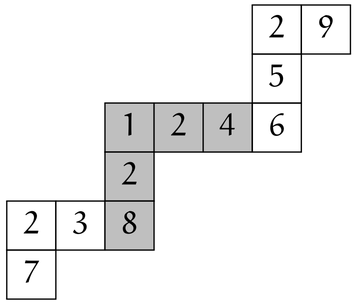
Proof. Let k be the number of LU corners of $RD(P_n)$. We show that

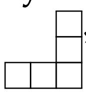
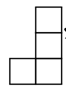
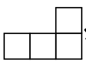
$$[q^{|E|}M_{(k+1,1^{n-k-1})}]Q = 0 \quad \text{but} \quad [q^{|E|}M_{(1,k+1,1^{n-k-2})}]Q > 0.$$

For a proper coloring with $|E|$ ascents and content $(k+1,1^{n-k-1})$, each copy of color 1 must be placed in an LU corner to respect the increasing conditions on the rows and columns. There are $k < k+1$ LU corners, so such a coloring does not exist, hence $[q^{|E|}M_{(k+1,1^{n-k-1})}]Q = 0$.

On the other hand, let T be one of the sub-ribbons in the statement. Color the LU corner of T with color 1 and the two neighboring boxes with color 2. Color the remaining $k-1$ LU corners of $RD(P_n)$ with 2, and arbitrarily color the rest of $RD(P_n)$ with the remaining distinct colors $\{3,4,\ldots,n-k\}$ such that the rows are increasing left to right and columns are increasing top to bottom. This is a proper coloring with a maximum ascent number by construction. Thus, $[q^{|E|}M_{(1,k+1,1^{n-k-2})}]Q > 0$, giving the desired conclusion. □

To illustrate the proof of Proposition 4.6, consider the following ribbon tableau with $k = 3$ LU corners. The given coloring, with the $(1,1,3)$ sub-ribbon shaded, shows that $[M_{(1,4,1^7)}]Q > 0$.

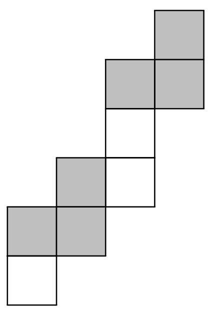


4.7. **Corollary.** *Suppose that P_n is a labeled path with vertex set $[n]$. If $RD(P_n)$ contains a $(3,1,1)$ sub-ribbon,* *, or begins with a $(2,1,1)$ sub-ribbon,* *, or ends with a $(3,1)$ sub-ribbon,* *, has a nonsymmetric CQF.*

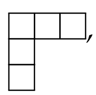
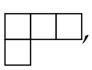
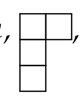
Proof. The result follows by combining Proposition 4.6 and Proposition 4.1. □

4.8. **Definition.** Suppose that P_n is a labeled path. We say $RD(P_n)$ is *regular* if $RD(P_n)$ has a row of length 2 followed by a row of length at least 2, or a terminal row of length 1. In this case, the $(2,1)$ sub-ribbon contained in these rows is called the *regular* $(2,1)$ *sub-ribbon*.

For example, the ribbon diagram below is regular, with two examples of regular $(2,1)$ sub-ribbons shaded.



4.9. **Proposition.** *Suppose that P_n is a labeled path such that*

- $RD(P_n)$ *does not contain a $(1,1,3)$ sub-ribbon,* *, does not begin with the $(1,3)$ sub-ribbon,* *, does not end with the $(1,1,2)$ sub-ribbon,* *, and*
- $RD(P_n)$ *is regular.*

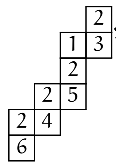
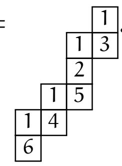
Then, $Q = X(P_n; \mathbf{x}, q)$ is not symmetric.

Proof. Let k be the number of LU corners of $RD(P_n)$. We show that

$$[q^{|E|}M_{(k,1^{n-k})}]Q > [q^{|E|}M_{(1,k,1^{n-k-1})}]Q.$$

Let A be the set of proper colorings of $RD(P_n)$ with content $(k, 1^{n-k})$ and maximum ascent number $|E|$, whose cardinality equals the coefficient $[q^{|E|}M_{(k,1^{n-k})}]Q$. These colorings use k copies of color 1 and each of the next $n-k$ colors once. Let B denote the set of proper colorings of $RD(P_n)$ with content $(1, k, 1^{n-k-1})$ and maximum ascent number $|E|$, whose cardinality equals the coefficient $[q^{|E|}M_{(1,k,1^{n-k-1})}]Q$. These colorings use k copies of color 2 and each of the colors $\{1, 3, 4, ..., n-k-1\}$ once.

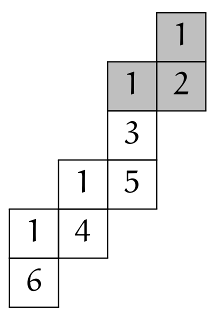
To show that $|B| < |A|$, we produce an injective map $\psi \colon B \to A$ that is *not* surjective. Define $\psi \colon B \to A$ as follows: given $T \in B$, $\psi(T)$ is obtained by replacing all 2's in the LU corners of T by 1's. For example, if

$T =$ , then $\psi(T) =$ .

By the first hypothesis, each $T \in B$ contains $k-1$ LU corners colored 2 and one LU corner colored 1. As a result, ψ is well-defined. Indeed, $\psi(T)$ has the maximum ascent number because replacing 2s with 1s maintains all ascents. Furthermore, ψ is an injection since the *unique* 2 *in* $\psi(T)$ *is adjacent to exactly one* 1, which uniquely determines T.

However, ψ is not a surjection. By the second hypothesis, $RD(P_n)$ contains a regular $(2, 1)$ sub-ribbon S. We construct a ribbon tableau $T' \in A$ not in the image of ψ: color the LU corners of S with color 1, and its RL corner with color 2. Color the remaining LU corners with color 1 and the rest of T' arbitrarily so that row and column entries strictly increase. Then T' is *not* in the image of ψ, since no coloring in $\psi(B)$ *contains a* 2 *that is adjacent to two* 1*'s* because the purported pre-image would not be a proper coloring, as it would have two adjacent boxes colored 2. □

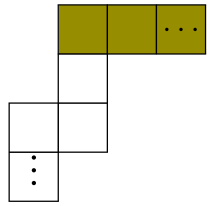
Below is an example of such a ribbon tableau T', with S shaded.



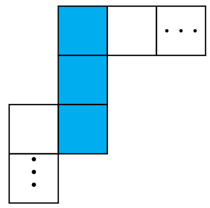
We now give a proof of the main theorem.

Proof of Theorem 1.1. Since P_n with the natural labeling has a symmetric CQF [SW16], we only need to prove the converse.

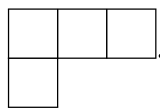
Let P_n be a path whose labeling is not natural. By Proposition 4.2, we can assume that P_n has the same number of LU and RL corners. Accordingly, $RD(P_n)$ has at least a stack of two rows or a stack of two columns. If $RD(P_n)$ consists entirely of a stack of rows or a stack of columns, $X(P_n; \mathbf{x}, q)$ is not symmetric by Proposition 4.4 and Corollary 4.5. Assume $RD(P_n)$ has a mix of stacks of rows and stacks of columns. By Proposition 4.1, we may assume that $RD(P_n)$ starts with a stack of columns (including, possibly, columns with two boxes). Consider the first transition point from the stack of columns to the stack of rows, where we must necessarily encounter a row $\mathbf{r}$ of length ≥ 3, colored in olive below.



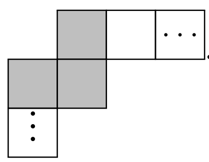
If the column left adjacent to **r** has at least 3 boxes (colored in cyan below), then $\mathrm{RD}(P_n)$ has a $(1, 1, 3)$ sub-ribbon and $X(P_n; \mathbf{x}, q)$ is not symmetric by Proposition 4.6.



Thus, we may assume that the column **c** left adjacent to **r** has 2 boxes. If **c** is the first column of $\mathrm{RD}(P_n)$, then the tableau begins with:



In this case, we can apply Proposition 4.6 to deduce that $X(P_n; \mathbf{x}, q)$ is not symmetric. Otherwise, there is at least one more column to the left of **c**. In this juncture, we have a regular $(2, 1)$ sub-ribbon:



Now, either $\mathrm{RD}(P_n)$ contains a $(1, 1, 3)$ sub-ribbon, begins with a $(1, 3)$ sub-ribbon, or ends with a $(1, 1, 2)$ sub-ribbon, so we are done by Proposition 4.6, or it does not, so we are done by Proposition 4.9. Hence, we conclude in all possible cases that $X(P_n; \mathbf{x}, q)$ is not symmetric. □

5. THE CHROMATIC QUASISYMMETRIC FUNCTION OF OTHER TREES

Theorem 1.1 shows that only the natural labeling of the path graph P_n results in a symmetric CQF. However, it is not the case that every tree will have some labeling that results in a symmetric CQF, as shown by the following theorems.

Consider the *star graph* $K_{1,n-1}$ on $n \geq 3$ vertices. The *central vertex* is the unique vertex of degree > 1.

5.1. **Proposition.** *The CQF of a (labeled) star graph G with n vertices is palindromic if and only if n is odd and the central vertex is labeled $\frac{n+1}{2}$.*

Proof. Suppose the central vertex of G is labeled j where $1 \leq j \leq n$. Consider the coefficients of the two monomial quasisymmetric functions $M_{(1,n-1)}$ and $M_{(n-1,1)}$ in $X(G; x, q)$. First note that they each consist of only one term, q^r and q^s, respectively. This is because there is only one way to color the vertices with content $(1, n-1)$, namely, the central vertex receives color 1 and other vertices receive color 2; the ascent number for this coloring is $r = n - j$. Similarly, there is only one way to color the vertices with content $(n-1, 1)$: the central vertex receives color 2, and other vertices receive color 1; the ascent number for this coloring is $s = j - 1$. Note that $r + s = n - 1$.

If n is even, then $r \neq s$, and $X(K_{1,n-1}; \mathbf{x}, q)$ is not palindromic. If n is odd, and $r \neq s$, then $X(K_{1,n-1}; \mathbf{x}, q)$ is not palindromic. Thus, $r = s = \frac{n-1}{2}$ is a necessary condition for $X(K_{1,n-1}; \mathbf{x}, q)$ to be palindromic. Note that $r = s = \frac{n-1}{2}$ if and only if the central vertex is labeled $\frac{n+1}{2}$. In this case, the labeled graph is invariant under the flip map of Definition 2.1 and thus, the coefficient of q^i is equal to that of q^{n-1-i} and $X(K_{1,n-1}; \mathbf{x}, q)$ is palindromic. □

5.2. **Proposition.** *The CQF of a (labeled) star graph G with $n \geq 4$ vertices is never symmetric.*

Proof. Symmetric CQFs are palindromic by [SW16, Corollary 2.8]. Thus, by Proposition 5.1, we need only show that the star graph on n vertices (for n odd) and central vertex labeled $\frac{n+1}{2}$ does not have a symmetric CQF. Using the content $(1, 2, n-3)$, the maximum ascent number is $\frac{n-1}{2}$; indeed, with the central vertex colored 1, there is an ascent for each edge $e = (\frac{n+1}{2})i$ with vertex $i > \frac{n+1}{2}$. However, for the content $(2, 1, n-3)$, the maximum ascent number is $\frac{n-1}{2} + 2$. This ascent number is achieved by coloring the central vertex with 2, and two vertices with labels smaller than $\frac{n+1}{2}$ with color 1, obtaining two additional ascents. Thus, the coefficient of $M_{(1,2,n-3)}$ is a polynomial in q of degree $\frac{n-1}{2}$, but the coefficient of $M_{(2,1,n-3)}$ is a polynomial in q of degree $\frac{n-1}{2} + 2$, and so $X(K_{1,n-1}; \mathbf{x}, q)$ is not symmetric. □

We also have the following general result on bipartite graphs.

5.3. **Proposition.** *Let G be a connected labeled bipartite graph with an odd number of edges and unequal bipartition, that is, $G = A \cup B$ for two independent sets A and B with $|A| \neq |B|$. Then, the CQF of G is not palindromic, and in particular, is not symmetric.*

Proof. Let $a = |A|$ and $b = |B|$; it is given that $a \neq b$. The coefficient of $M_{(a,b)}$ in $X(G; \mathbf{x}, q)$ consists of a single term q^r, where r equals the number of edges ij such that $i \in A$ and $j \in B$ with $i < j$. Similarly, the coefficient of $M_{(b,a)}$ in $X(G; \mathbf{x}, q)$ consists of a single term q^s, where s equals the number of edges ij such that $i \in A$ and $j \in B$ with $i > j$. Since $r + s = |E(G)|$ is odd, it follows that $r \neq s$. As the coefficients of $M_{(a,b)}$ and $M_{(b,a)}$ differ, $X(G; \mathbf{x}, q)$ is not palindromic. □

5.4. **Corollary.** *Let T be a labeled tree with an even number of vertices with unequal bipartition. Then, the CQF of T is not palindromic, and in particular, is not symmetric.*

With this in mind, we therefore conclude with the following question, which was recently answered in [GPS24].

5.5. **Question.** For which labeled trees T is $X(T; \mathbf{x}, q)$ symmetric?

In [GPS24], the authors prove that the path graph P_n with the natural labeling is the *only* tree with symmetric CQF.

The reader may wonder how our paper and [GPS24] differ and coincide. As noted earlier, both papers contain a version of Proposition 4.2. Our map in the proof of Proposition 4.9 is a specialization of [GPS24, Definition 4.7] with $k = 1$. Beyond that, the proof methods are different, and all results were arrived at independently. It is interesting that in both papers, enumerating the multiplicity of $q^{|E|}$ in a CQF is used to determine whether a CQF is symmetric.

6. Statements and Declarations

Conflict of Interest Statement. The authors collectively declare that there is no conflict of interest arising from this work.

Data Availability Statement. No data sets were used in the current manuscript.

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