FUNCTIONS OF SEV. VAR'S 14.1

Quadric surfaces are 3-D surfaces, but not necessarily 'functions'

$z = f(x,y)$ ← height in terms of x-y coord

$w = f(x,y,z)$ ← Temp. (e.g.) in terms of x, y, z coord.

LEVEL CURVE / SURFACE

Set a function to a constant output + get a shape or a surface

#1 $f(x,y) = \frac{y}{x^2}$ if $f(x,y) = c \Rightarrow y = cx^2$ a parabola at height c

#2 $f(x,y,z) = x^2+y^2+z^2$ if $f(x,y,z) = c = x^2+y^2+z^2$ a sphere of radius $\sqrt{c}$

LIMITS 14.2

If $f(x,y) \to L$ as $(x,y) \to (x_0,y_0)$ NO MATTER WHICH PATH ONE USES TO APPROACH (x_0,y_0), The L is limit of $f(x,y)$ at (x_0,y_0)

#1 $\lim\limits_{(x,y)\to(-1,2)} \frac{xy+y-2x-2}{x+1} \left(\frac{0}{0}\right) = \lim \frac{\cancel{(x+1)}y - 2\cancel{(x+1)}}{\cancel{x+1}}$

$= \lim\limits_{y\to 2} y-2 = 0$ (NO PROBLEMS)

CONTINUITY

Concept similar to that in 2D. However, in 3D, need to check multiple approaches to a pt to ensure limit exists.

$f(x,y) = \begin{cases} \frac{x^2}{x^2+y} & (x,y) \neq (0,0) \\ 0 & (0,0) \end{cases}$

If $y=0$, $f(x,0) = \frac{x^2}{x^2} = 1$

$\Rightarrow$ as $(x,0) \to (0,0)$, $f(x,y) = 1$

If $y = x^2$, $f(x,x^2) = \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$

$\Rightarrow$ as $(x,x^2) \to (0,0)$, $f(x,y) = \frac{1}{2}$

$\therefore$ since $1 \neq \frac{1}{2} \neq 0$ NO limit exist and $f(x,y)$ NOT CONT'S at $(0,0)$

PARTIAL DERIV'S 14.3

all variables are considered constants except for the one being differentiated with respect to. We use partial d symbol ∂.

#1 $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

$$\frac{\partial f}{\partial x} = 5y - 14x + 3 = \underline{f_x} \qquad \frac{\partial f}{\partial y} = 5x - 2y - 6 = \underline{f_y}$$

#2 $xy + z^3x - 2yz = 0$ (NEED TO USE IMPLICIT DIFF'N)

want $\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}$ (implicit diff wrt x)

$$y + \left(3z^2\frac{\partial z}{\partial x}\right)x + z^3 - 2y\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x}(3z^2x - 2y) = -y - z^3$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\left.\frac{y+z^3}{3z^2x-2y}\right|_{(1,1,1)} = -\frac{1+1}{3\cdot1-2\cdot1} = -\frac{2}{3-2} = \underline{-2}$$

$\frac{\partial f}{\partial x}$ indicates slope moving in x-direction

$\frac{\partial f}{\partial y}$ indicates slope moving in y-direction.

HIGHER DERIVS 14.3

Given $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ we can find $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ as in 2D.

We also have "mixed" 2nd derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

If f is cont's around (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$

EX $\quad f(x,y) = e^x \ln(3-y^2)$

$f_x = e^x \ln(3-y^2)$

$f_{xx} = e^x \ln(3-y^2) = f(x,y)$

$f_{xy} = \dfrac{-2e^x y}{3-y^2}$

$f_y = e^x\left(\frac{1}{3-y^2}\right)(-2y) = \dfrac{-2e^x y}{3-y^2}$

$$f_{yy} = \frac{(3-y^2)(-2e^x) + 2e^x y(-2y)}{(3-y^2)^2}$$

$$= \frac{-6e^x + 2e^x y^2 - 4e^x y^2}{(3-y^2)^2}$$

$$= \frac{-6e^x - 2e^x y^2}{(3-y^2)^2}$$

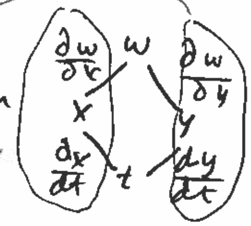
$$f_{yx} = \frac{-2e^x y}{3-y^2}$$

CHAIN RULE 14.4

2 var's $w = f(x,y)$ where x, y are functions of t

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

can use diagram



3 var's $w = f(x,y,z)$ where x, y, z are all functions of t

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

2 var's $w = f(x,y)$ with x, y functions of 2 var's r, s

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \quad \text{etc.}$$

EX. $f(x,y) = x^2 + y^2 \quad x = r - s \quad y = r + s$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = (2x)(1) + (2y)(1) = 2(r-s) + 2(r+s) = \underline{4r}$$

DIRECTIONAL DERIV'S 14.5

We can use $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to find slope (deriv) in <u>any</u> direction.

Let $\vec{u} = a\vec{i} + b\vec{j}$ be a <u>unit</u> vector $(a^2 + b^2 = 1)$

then $D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$ is the deriv in direction $\vec{u}$

For angle θ, $\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ is a <u>unit</u> direction vector.

$$\therefore\ D_\theta f = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

The vector of all partial derivatives of $w = f(x,y,z)$ is called the GRADIENT

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\therefore\ D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

EX Given $f(x,y,z) = x^3 - xy^2 - z$

find deriv of f in direction of $\vec{A} = 2\vec{i} - 3\vec{j} + 6\vec{k}$ at $P_0(1,1,0)$

① $\nabla f = \begin{pmatrix} 3x^2 - y^2 \\ -2xy \\ -1 \end{pmatrix} \Rightarrow \nabla f\Big|_{(1,1,0)} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

② Need unit direction vector

$$\vec{u} = \frac{2\vec{i} - 3\vec{j} + 6\vec{k}}{\sqrt{4+9+36}} = \frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$$

③ $D_{\vec{u}} f\Big|_{(1,1,0)} = \nabla f\Big|_{(1,1,0)} \cdot \vec{u} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2/7 \\ -3/7 \\ 6/7 \end{pmatrix} = \frac{4}{7} + \cancel{\frac{6}{7}} \cancel{- \frac{6}{7}} = \underline{\frac{4}{7}}$

TANGENT PLANES + NORMAL LINES

Given a level surface (or curve) $f(x,y,z)=c$ and pt $P_0(x_0,y_0,z_0)$, the gradient of f evaluated at P_0, $\vec{\nabla} f\Big|_{P_0}$, is NORMAL to the surface at P_0

Therefore,

Tangent plane at P_0: $f_x\Big|_{P_0}(x-x_0)+f_y\Big|_{P_0}(y-y_0)+f_z\Big|_{P_0}(z-z_0)=0$

Normal line at P_0: $\dfrac{x-x_0}{f_x|_{P_0}}=\dfrac{y-y_0}{f_y|_{P_0}}=\dfrac{z-z_0}{f_z|_{P_0}}\ (=t)$

EG. $x^2+2xy-y^2+z^2=7$ and $P_0(1,-1,3)$

$\Rightarrow \vec{\nabla} f=\langle 2x+2y,\ 2x-2y,\ 2z\rangle \Rightarrow \vec{\nabla} f\Big|_{P_0}=\langle 0,4,6\rangle$

$\therefore$ tangent plane: $4(y+1)+6(z-3)=0 \Rightarrow \underline{2y+3z=7}$

Normal line:

$$\left.\begin{array}{lll} x-1=0\cdot t & \Rightarrow & x=1 \\ \dfrac{y+1}{4}=t & \Rightarrow & y=-1+4t \\ \dfrac{z-3}{6}=t & \Rightarrow & z=3+6t \end{array}\right\}$$

14.6 Tangent planes + Linear approx p853
14.7 MAX/MIN/Saddle pts p863
14.8 Lagrange Multiplier p872

14.6 <u>Tangent plane for $z = f(x,y)$ at $P_0(x_0, y_0, f(x_0,y_0))$</u>

$$f_x\Big|_{P_0}(x-x_0) + f_y\Big|_{P_0}(y-y_0) - (z-z_0) = 0$$

Linear approx for $z = f(x,y)$ at (x_0, y_0)

$$L(x,y) = f(x_0,y_0) + f_x\Big|_{P_0}(x-x_0) + f_y\Big|_{P_0}(y-y_0)$$

14.7 <u>MAX/MIN/Saddle</u>

Theory similar to 2D case ① look for pts where $f_x = 0$, $f_y = 0$ (critical pts)

② compute discriminant $D(x,y) = f_{xx} f_{yy} - f_{xy}^2$

Given (a,b) a critical pt.

$D(a,b) > 0$ $f_{xx}(a,b) < 0$ $\Rightarrow$ (a,b) local max
$D(a,b) > 0$ $f_{xx}(a,b) > 0$ $\Rightarrow$ (a,b) local MIN
$D(a,b) < 0$ $\Rightarrow$ (a,b) saddle pt
$D(a,b) = 0$ inconclusive.

EG. p870 #9 $f(x,y) = x^2 - y^2 - 2x + 4y + 6$ $f_x = 2x - 2 \Rightarrow x = 1$ $f_y = -2y + 4 \Rightarrow y = 2$
$f_{xx} = 2$ $f_{yy} = -2$ $f_{xy} = 0$
$\Rightarrow D(x,y) = 2(-2) - 0^2 = -4 < 0$ $\therefore$ <u>$(1,2)$ is saddle pt.</u>

14.8 <u>Lagrange Multiplier</u>

Suppose want to minimize $f(x,y,z)$ subject to constraint $g(x,y,z) = 0$
We use Lagrange Mult <u>$\nabla f = \lambda \nabla g$</u> to get add'l eqs to find values.

EG. p879 #4 Find extreme values of $f(x,y) = x^2 y$ on line $x + y = 3$

$\Rightarrow g(x,y) = x + y - 3 = 0$

$$\nabla f = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \left.\begin{matrix} 2xy = \lambda \\ x^2 = \lambda \end{matrix}\right\} \Rightarrow 2xy = x^2 \Rightarrow 2xy - x^2 = 0$$
$$\Rightarrow x(2y - x) = 0$$

$\Rightarrow x = 0 \Rightarrow 0 + y - 3 = 0 \Rightarrow y = 3 \Rightarrow$ pt $(0,3)$

or $\Rightarrow 2y = x \Rightarrow 2y + y - 3 = 0 \Rightarrow 3y = 3 \Rightarrow y = 1$
$\Rightarrow x + 1 - 3 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$ $\Big\} \Rightarrow$ pt $(2,1)$

$f(0,3) = 0^2 \cdot 3 = 0$ ← min
$f(2,1) = 2^2 \cdot 1 = 4$ ← max