EQUATIONS OF SETS OF POINTS 11.6

We will use the distance formula to derive various equations of curves (= sets of points = loci). $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{a^2+b^2}$

Usually, we need to carefully examine the implications of given data before using the formula, however.

EG Find the equation for the set of points equi-distant from the line $x=2$ and the origin.

NOTE: The origin is a fixed point, $(0,0)$.
The line, $x=2$, consists of multiple points, each having the form $(2,y)$.

$\therefore$ distance from a point on our curves, $P(x,y)$, to the origin is: $\sqrt{(x-0)^2+(y-0)^2} = d_1$

BUT
distance from a point on our curve, $P(x,y)$, to the line $x=2$ is: $\sqrt{(x-2)^2+(y-y)^2} = d_2$

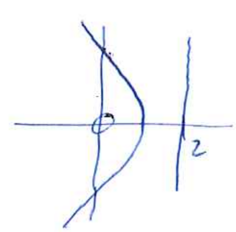
We are told these distances are equal.

$d_1 = \sqrt{x^2+y^2}$ and $d_2 = \sqrt{(x-2)^2}$

Squaring and setting equal,

$$x^2+y^2 = (x-2)^2 = x^2 - 4x + 4$$

or $$y^2 = -4x + 4$$

(parabola )

$\Rightarrow$ NOTE: The distance formula will be used to derive equations of the "conic sections"

CIRCLES (AP-13)

Def. A circle is the set of points in a plane whose distance from a fixed point in the plane is a constant.

Def The fixed point is called the center of the circle

Def The fixed constant distance is called the radius of the circle.

To obtain a general equation for a circle we use the distance formula.

Let (x, y) be an arbitrary point of the circle.
Let (h, k) be the fixed point – the center.
Let a be the constant distance – the radius.

Then we obtain

$$\sqrt{(x-h)^2+(y-k)^2} = a$$

or

$$(x-h)^2+(y-k)^2 = a^2$$

EX

① rad $= 2$, center $= (3,4)$

$$\Rightarrow (x-3)^2+(y-4)^2 = 2^2 = 4$$

② $(x+2)^2 + (y-1)^2 - 5 = 0$

$$\Rightarrow (x-(-2))^2 + (y-1)^2 = (\sqrt{5})^2$$

$\Rightarrow$ rad $= \sqrt{5}$ center $= (-2, 1)$

③ rad $= a$, center $= (0,0)$ (origin)

$$x^2+y^2 = a^2$$

NOTE:

circle at origin $x^2+y^2=a^2$

circle at (h,k) $(x-h)^2+(y-k)^2=a^2$

We can see the general equation as a double shift of the origin centered equation.

$x \to x-h$ shifts the circle right h units.

$y \to y-k$ shifts the circle up k units.

EX B Find the equation of a circle with center (2,2) going through (4,5).

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-2)^2 + (y-2)^2 = r^2$$

$$(4-2)^2 + (5-2)^2 = r^2$$

$$2^2 + 3^2 = r^2$$

$$4 + 9 = 13 = r^2$$

$\therefore$ the equation is

$$(x-2)^2 + (y-2)^2 = 13$$

or

$$x^2 - 4x + 4 + y^2 - 4y + 4 = 13$$

$$x^2 - 4x + y^2 - 4y = 5$$

PARABOLAS (p 683)

Def A parabola is the set of points in a plane equidistant from a given fixed point and a fixed line in a plane.

Def. The fixed point is called the focus of the parabola.

Def The fixed line is called the directrix of the parabola.

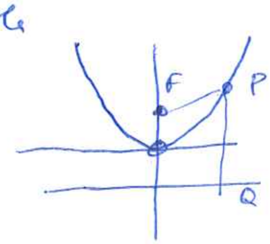
Def The pt midway between the focus and the directrix is called the vertex

As before, we use the distance formula.

Let (x, y) be an arbitrary point on the parabola

Let $y = -p$ be the fixed line.

Let $(0, p)$ be the focus.



$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}$$

Equating, squaring and simplifying yields

$$y = \frac{1}{4p} x^2$$

Similarly

$$y = -\frac{1}{4p} x^2$$

$$x = \frac{1}{4p} y^2$$

$$x = -\frac{1}{4p} y^2$$

Using shifting procedures as before.

$$y - k = \frac{1}{4p} (x-h)^2$$

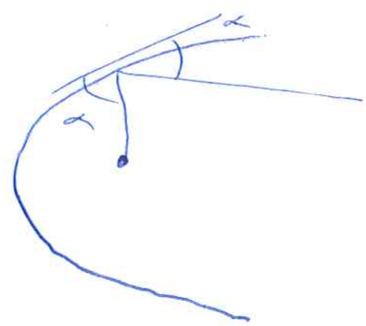
is a parabola shifted right h units and up k units and (h, k) is the vertex

The equation for a parabola is quadratic in either x or y, but not both. In general,

$$y = ax^2 + bx + c$$

is a parabola.

NOTE.



Parabolas reflect "rays of light" toward the focus.

p691 q#59

Analyze $x^2+8y-2x=7$

$\Rightarrow x^2-2x\underline{+1} = -8y+7\underline{+1} = -8y+8$

$(x-1)^2 = -8(y-1)$

$= -4(2)(y-1)$

$\Rightarrow$ x^2 tells us this is a <u>vertical</u> par.

$-4py \Rightarrow$ concave DOWN

$\therefore$ Vertex $(1,1)$ $p=2$

Axis of sym. $x=1$

Focus $(1, 1-2) = (1,-1)$

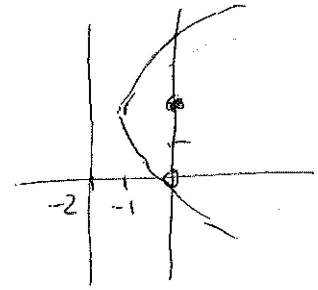
Directrix is $y=3$

$V\ (h,k)$

$F\ (h, k\pm p)$

$D:\ y=k\mp p$

E6 ① Given $(0,2)$ as a focus and $x=-2$ as the directrix find the equation for the parabola.



$F(0,2)$, $D: x=-2 \Rightarrow p=1$

$\Rightarrow V(-1,2)$

$\Rightarrow (y-2)^2 = 4(1)(x+1)$

② Given $y = \dfrac{x^2}{16}$ find the focus and directrix.

$\Rightarrow 16y = x^2$

$\Rightarrow x^2 = 4(4)y$

$\Rightarrow p = 4$

$\therefore$ Focus is $(0,4)$

Directrix is $y=-4$