PARAMETRIC EQUATIONS (11.1)

When a particle moves through space, its movement is sometimes described by 2 equations:
- one describing its x-direction motion,
- The other describing its y-direction motion,

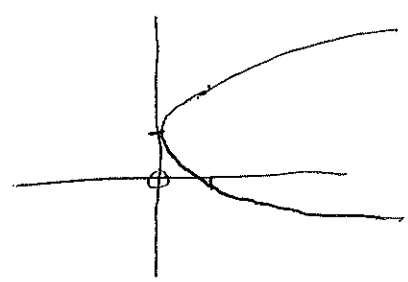
with both given in terms of time, t.

I.e. $x = f(t)$, $y = g(t)$

p653 Such equations are called <u>parametric</u> with the <u>parameter</u> t.

p654 <u>EX 1</u> $x = t^2$, $y = t + 1$

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



<u>NOTE</u>: We have <u>one</u> curve, but it takes 2 equations to describe it.

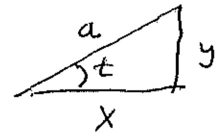
⇒ Sometimes, we can eliminate the parameter, but we may lose information.

$y = t + 1 \Rightarrow t = y - 1 \Rightarrow \underline{x = (y-1)^2}$

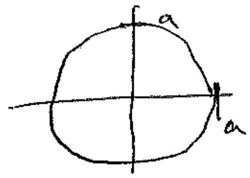
↑ DOES NOT indicate "when" x, y are at certain values

EX 3b

$x = a\cos t \quad y = a\sin t$



$\Rightarrow$



$\Leftrightarrow \quad x^2 + y^2 = a^2$

EX 4

$x = \sqrt{t} \qquad y = t \qquad t \geqslant 0$

Find Cartesian equation.

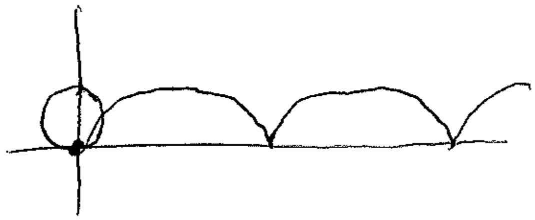
$y = t = (\sqrt{t})^2 = x^2$

$\Rightarrow \quad y = x^2$

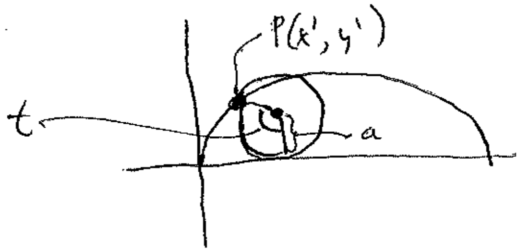
BUT, given the condition $t \geqslant 0$ (to avoid any problem with $x = \sqrt{t}$), x cannot be negative.

CYCLOID (p 657)

The *cycloid* is the curve which describes the path of a point on the *edge* of a rolling circle.



We can describe this curve via parametric equations in terms of the angle of rotation t of the circle and using a, the radius of the circle.



Let x', y' be the coordinates of P relative to the center of the circle.

Assuming t is in radians, the distance along the circumference corresponding to angle t is merely at. (I.e. a full circle has angle $t = 2\pi$ and the full circumference is $2\pi a$, since a is the radius.)

Therefore after a movement of t, the center of the circle is at (at, a)

The coordinates of $P(x, y)$ relative to the origin are

$$x = x' + at$$
$$y = y' + a$$

By previous example,

$$x' = a\cos\theta \qquad y' = a\sin\theta \qquad \text{for some } \theta.$$

By analysis $\theta = \frac{3\pi}{2} - t$

Thus $\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t$

and $\sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$

$$\therefore \quad x = at - a\sin t = a(t - \sin t)$$
$$y = a - a\cos t = a(1 - \cos t)$$

These form the parametric equation of a cycloid.

CALCULUS AND PARAMETRIC CURVES (11.2)

P 661

Given $x = f(t)$, $y = g(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Remember the two equation give one curve, so it makes sense to try to find some way to find the slope of the curve.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

EX Given $x = 3t+1$ $\quad y = t^2$

$$\Rightarrow \frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 2t$$

$$\therefore \frac{dy}{dx} = \frac{2t}{3}$$

EX 1 $x = \sec t$ $\quad y = \tan t$ $\qquad$ tangent at $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$

$$\frac{dx}{dt} = \sec t \tan t \qquad \frac{dy}{dt} = \sec^2 t$$

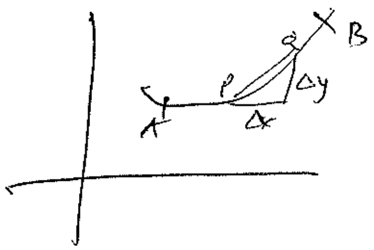
$$\frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}\Bigg|_{t=\pi/4} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\therefore \frac{y-1}{x-\sqrt{2}} = \sqrt{2} \Rightarrow y - 1 = \sqrt{2}x - 2$$

$$\Rightarrow \underline{y = \sqrt{2}x - 1}$$

LENGTH OF PARAMETRIC CURVE

p 665-7



$$|PQ| = \sqrt{\Delta x^2 + \Delta y^2}$$

$\therefore$ Length from A to B $\doteq \sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$

Exact length from A to D $= \int \sqrt{dx^2 + dy^2}$

If $y = f(x)$ we get

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If $y = f(t)$, $x = g(t)$ we have

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EX p669 (11.2) #26

$x = t^3 \qquad y = \frac{3t^2}{2} \qquad t = 0 \to \sqrt{3}$

$\frac{dx}{dt} = 3t^2 \quad \frac{dy}{dt} = 3t$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^4 + 9t^2 = 9t^2(t^2+1)$

$\therefore L = \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt$

$u = t^2+1$

$\frac{du}{dt} = 2t$

$\frac{du}{2} = t\,dt$

$= \frac{3}{2}\int_{t=0}^{\sqrt{3}} u^{1/2}\, du$

$= \cancel{\frac{3}{2}} \frac{u^{3/2}}{\cancel{3/2}} \Big]_{t=0}^{\sqrt{3}} = (t^2+1)^{3/2} \Big|_0^{\sqrt{3}}$

$= (3+1)^{3/2} - 1^{3/2} = 4^{3/2} - 1 = 2^3 - 1$

$= 8 - 1 = \underline{7}$

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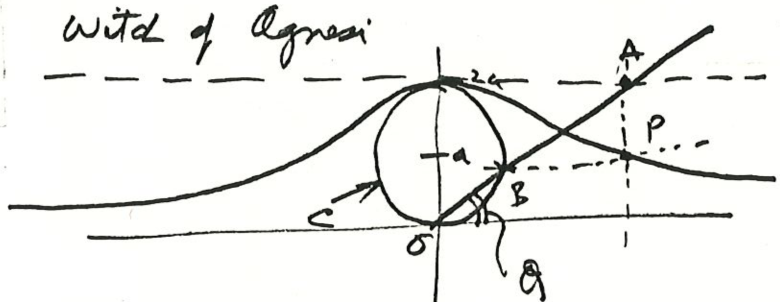
11-5

Witch of Agnesi



C circle rad a cent (0, a)

P has y-coord of B, x-coord of A.

(1) Need to find coord of B - find intersect of line OA and circle.

eq of line: $y = \tan\phi\, x \quad \Rightarrow \quad x = \dfrac{y}{\tan\phi}$

eq of circle $x^2 + (y-a)^2 = a^2$

sub $\dfrac{y^2}{\tan^2\phi} + y^2 - 2ay + \cancel{a^2} = \cancel{a^2}$

$y^2 + y^2\tan^2\phi - 2ay\tan^2\phi = 0$

$y^2(1+\tan^2\phi) - 2ay\tan^2\phi = 0$

$\therefore y = 0$ or

$y(\sec^2\phi) - 2a\tan^2\phi = 0$

$\therefore y = \dfrac{2a\tan^2\phi}{\sec^2\phi} = \dfrac{2a\sin^2\phi}{\cancel{\cos^2\phi}\cdot\frac{1}{\cancel{\cos^2\phi}}} = 2a\sin^2\phi$ = y coord of B = y coord of P

optional

(2) Need x-coord of pt A

$x_A = (\cos\phi)\,\overline{OA}$

But also $y_A = 2a = \overline{OA}\sin\phi$

$\therefore \sin\phi = \dfrac{2a}{\overline{OA}} \qquad \therefore \overline{OA} = \dfrac{2a}{\sin\phi}$

$x_A = \cos\phi \cdot \dfrac{2a}{\sin\phi} = \underline{2a\cot\phi}$

= x coord of P.

∴ parametric equations of witch are

$x = 2a\cot\phi$

$y = 2a\sin^2\phi$

(usual cartesian eq $y = \dfrac{8a^3}{x^2+4a^2}$)