THE BINOMIAL SERIES (10.10) p638

The Taylor (Maclaurin) series for $f(x) = (1+x)^m$ is called the Binomial Series. If m is an integer ≥ 0, the series is finite and has $m+1$ terms. Otherwise it is infinite

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots$$

$$\ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \ldots$$

standard abbreviation for constant coefficients:

$$\binom{m}{1} = m$$

$$\binom{m}{2} = \frac{m(m-1)}{2!}$$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}$$

This series will converge if $|x| < 1$

We can also write the binomial series as

$$(1+x^m) = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

cf. <u>EX</u> 2 p 640

In 3.11, we saw that $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for $|x|$ small.

Using binomial series, we have

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

Problem: estimate $\sqrt{1.25}$

$$\sqrt{1.25} = \left(1+\frac{1}{4}\right)^{1/2}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{8}\left(\frac{1}{4}\right)^2 + \frac{1}{16}\left(\frac{1}{4}\right)^3 - \cdots$$

$$= \underbrace{1 + \frac{1}{8} - \frac{1}{128}} + \frac{1}{1024}$$

$$= 1.117$$

SOME OTHER "PROBLEMATIC" FUNCTIONS (for calculators):

LOGS

p. 640 By using the Taylor series with $a=0$, we obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + \dots$$

(NOTE: no factorials).

This expansion is verified also by integration:

$$\int_0^x \frac{dt}{1+t} = \int_0^x (1 - t + t^2 - t^3 + \dots)\,dt$$

PI = π

NOTE: $\pi \neq \frac{22}{7}$ but $\underline{3.14}15926\dots$ is close to $\underline{3.14}28571$

One way to obtain a "formula" for π is to use the Taylor series for $\underline{\arctan x}$, since $\arctan 1 = \frac{\pi}{4}$.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Thus $\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$

p 642 This is known as Leibniz's formula.

Unfortunately, this converges very slowly.

The Taylor series converges more rapidly as x is closer to 0. Thus, a better way to calculate π is to use a combination of smaller angles. For example, let

$$\alpha = \arctan \frac{1}{2} \qquad \text{and} \qquad \beta = \arctan \frac{1}{3}$$

then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \frac{5/6}{5/6} = 1 = \tan\frac{\pi}{4}$$

i.e.

$$\frac{\pi}{4} = \alpha + \beta = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

For even faster convergence, one can use

$$\pi = 48 \arctan\frac{1}{18} + 32 \arctan\frac{1}{57} - 20 \arctan\frac{1}{239}.$$

Table 10.1

cf pg 644 for a summary list of often-used Maclaurin series.

INDETERMINATE FORMS p642

Think back to Chapter I and the motivation for limits.
For example, given $\frac{x^2-1}{x-1}$, we can NOT find the value of this function at $x=1$ since both the top and bottom are 0.

Chapter 1 suggested we use "limits" and before taking the limit, perform algebraic simplification.

We could also use L'Hôpital's rule, repeating, if necessary, i.e.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{only if } f(a)=g(a)=0.$$

Eg

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = 2$$

or $$= \lim_{x\to 1} \frac{2x}{1} = 2$$

Sometimes, the differentiation can be difficult, especially if repeated. In these cases, using Taylor series can be helpful

p642 EX 5 $\lim_{x\to 1} \frac{\ln x}{x-1}$

① using L'Hôpital, $= \lim \frac{\frac{1}{x}}{1} = \underline{\underline{1}}$

② using Taylor series,

$$\ln x \text{ (with } a=1) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \cdots$$

$$\therefore \frac{\ln x}{x-1} = 1 - \frac{x-1}{2} + \cdots.$$

$$\therefore \lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1}\left(1 - \overbrace{\frac{x-1}{2} + \cdots}^{\to 0}\right) = \underline{\underline{1}}$$

EX 6 $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\Rightarrow \sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$\therefore \lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right) = \underline{\underline{-\frac{1}{2}}}$$

OTHER IMPLICATIONS

p 643-4

By using the Taylor expansion, we can see that

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

Thereby motivating the standard <u>definition</u> that

$$e^{i\theta} = \cos\theta + i\sin\theta$$