TAYLOR'S THEOREM (10.9) p 631

Thm 23 (p 631. (of Sec 4.2 p 232ff [illegible])

If f and its 1st n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a,b]$ and if $f^{(n)}$ is differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Corol 1 If f has derivatives of all orders in an open interval I containing a, then for each integer $n > 0$, and $\forall x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x,a)$$

where

$$R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \in (a,x)$$

This is called "Taylor's Formula". p 631

$R_n(x,a) = R_n(x)$ if a is obvious

USES

It is much easier to calculate polynomial functions than others. If we can obtain a "good" approximation to a transcendent function (e.g. e^x, $\sin x$) which is a polynomial, we can use the polynomial. How "good" the approximation is, will depend on the "remainder" term if we use a Taylor series.

EX 1 (p. 632)

Let $f(x) = e^x$ and use $a = 0$.

$$e^x = f(x) = f'(x) = f^{(n)}(x) \quad \forall n$$

$$\therefore \; f(0) = e^0 = 1 = f^{(n)}(0) \quad \forall n$$

$$\therefore \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + R_n(x, 0)$$

where $R_n(x,0) = \dfrac{e^c}{(n+1)!} x^{n+1}$ for some c between 0 and x.

NOTE: we would like to be sure that the Taylor series converges! We also would like to know how large/small the remainder is.

ESTIMATING THE REMAINDER p 633

Thm 24 If there are (positive) constants M and r s.t. $|f^{(n+1)}(t)| \le M r^{n+1}$ $\forall t$ between a and x inclusive, then

$$|R_n(x,a)| \le M \frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}.$$

FURTHERMORE, if these conditions hold in addition to the conditions of Taylor's Thm. w.r.t $f(x)$, then the Taylor Series actually does converge to $f(x)$.

EX 2 Let $f(x) = \sin x$.

$\therefore f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$

etc. $\therefore f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$

$\therefore$ by Taylor's Thm

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x,0).$$

Since $f^{(n)}(x) = \pm \sin x$ or $\pm \cos x$, $|f^{(n)}(x)| \le |1|$

Thus we can choose $M=1$ and $r=1$ in the Remainder Thm. and obtain

$$|R_{2k+1}(x,0)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}$$

But as $k \to \infty$, $\frac{|x|^{2k+2}}{(2k+2)!} \to 0$ and thus $R_{2k+1}(x,0) \to 0$.

Therefore the Maclaurin series converges for every x.

(skip 22-6 13th ed)

USING TAYLOR SERIES

EX 4

a) Using power series operations and known series, find Taylor series for:

$\frac{1}{3}(2x + x\cos x)$

Remember $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\therefore \quad x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots$

$\therefore \frac{1}{3}(2x + x\cos x)$

$= \frac{2x}{3} + \frac{x}{3} - \frac{x^3}{3\cdot 2!} + \frac{x^5}{3\cdot 4!}$

$= x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots$

ESTIMATING ERROR

p 637 #36 Estimate the error if:

$$P_4 \doteq 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

is use to estimate the value of e^x at $x = \frac{1}{2}$.

In Remainder Estimation Thm, we look at $n = 4$ and

$$|R_4(x)| \le M \frac{|x-0|^5}{5!}$$

Need to determine M s.t. $|f^{(5)}(t)| \le M$ for $t \in (0, x)$

But since $f(x) = e^x$, $f^{(5)}(x) = e^x$

Since $x = \frac{1}{2}$, $M = e^{1/2} \Rightarrow |e^t| \le e^{1/2}$ on $[0, 0.5]$

$$\therefore |R_4(x)| \le 1.6 \frac{|.5-0|^5}{5!}$$

$$= 1.6 \cdot \left(\frac{1}{32}\right)\left(\frac{1}{120}\right)$$

$$= \frac{1.6}{3840}$$

$$= .0004167$$

$$= 4.167 \times 10^{-4}$$

TRUNCATION ERROR

The Remainder Estimation Thm can be used to indicate how far a Taylor series must be expanded to give an answer with a small error.

EX Calculate e with error $< 10^{-6}$.

From EX 1 above, with $x=1$, we know

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n(1,0)$$

where $R_n(1,0) = e^c \frac{1}{(n+1)!}$ for $c \in (0,1)$

From 38, p185 we know $e < 3$. Therefore

$$\frac{1}{(n+1)!} < R_n(1,0) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!}$$

By trial and error, we find that $\frac{1}{9!} > 10^{-6}$ and $\frac{3}{10!} < 10^{-6}$

$\therefore$ $n+1$ should be at least 10 or n should be at least 9.

$\therefore$ for an error $< 10^{-6}$, we choose

$$e \doteq 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \doteq 2.718282$$