POWER SERIES (10.7)

p617 *Def* A **power series** about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

If this series is FINITE, i.e. if $a_k = 0$ for all $k >$ some N, the series is very easy to evaluate for a given value of x.

We would like to approximate a function $f(x)$ by a sequence of polynomials $\sum_{i=1}^{n} a_i x^i = f_n(x)$

This is done in one of 2 common ways – centered at $\underline{0}$, or centered at point $\underline{a}$.

MACLAURIN SERIES (10.8)

p627

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

$$= f(0) + f'(0)x + f''(0)\cdot\frac{x^2}{2!} + \dots + \frac{f^{(n)}(0)\,x^n}{n!} + \dots$$

TAYLOR SERIES

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$$

$$= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

TAYLOR POLYNOMIALS p628

In 3.11 we introduced the Linearization of a function f at a point a. This was merely the tangent line at x=0 to f. I.e. $L(x) = f(a) + f'(a)(x-a)$

cf p 203.

Using a Taylor series, we can generalize this approximation to a polynomial of any degree.

Def Let f be a function with derivatives of order K=1, 2, ... N in some interval containing pt a.
Then for any integer n between 0 and N, the Taylor polynomial of order n generated by f at x=a is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note that the linearization of f at a is merely the Taylor polynomial of order 1 of f at a.

EX 2 (p 628)

TAYLOR SERIES for $f(x)=e^x$ AT $x=0$
(i.e. MACLAURIN SERIES)

$$f(x) = e^x = f'(x) = f''(x) = f'''(x) = f^{(n)}(x)$$

$$f(0) = e^0 = 1 = f'(0) = f''(0) = f'''(0) = f^{(n)}(0)$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

EX 3 (p 629)

TAYLOR SERIES for $f(x) = \cos x$ AT $x=0$

$$f(x) = \cos x$$
$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{iv}(x) = \cos x = f(x)$$

In general, $f^{(2n)}(x) = (-1)^n \cos x$

$$f^{(2n+1)}(x) = (-1)^{n+1} \sin x$$

At $x=0$, $f^{(2n)}(0) = (-1)^n \cos 0 = (-1)^n$

$$f^{(2n+1)}(0) = (-1)^{n+1} \sin 0 = 0$$

Therefore, the Taylor series (Maclaurin series) for $\cos x$ (at $x=0$) is:

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots$$

$$= 1 + 0 \cdot x - 1 \cdot \frac{x^2}{2!} + 0x^3 + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

EXAMPLE

p 630 # 13

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f'(x) = -1(1+x)^{-2}$$

$$f''(x) = +2(1+x)^{-3}$$

$$f'''(x) = -3\cdot2(1+x)^{-4}$$

$$f^{iv}(x) = +4\cdot3\cdot2(1+x)^{-5}$$

In general, $f^{(n)}(x) = n!\,(1+x)^{-(n+1)}\,(-1)^n$

at $x=0$ $\quad f^{(n)}(0) = n!\,(1)^{-(n+1)}(-1)^n = n!\,(-1)^n$

Therefore, the Taylor (Maclaurin) series for $\frac{1}{1+x}$ (at $x=0$) is:

$$\frac{1}{1+x} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)\,x^n}{n!} + \dots$$

$$= 1 + (-1)x + \frac{2}{2!}x^2 - \frac{3!}{3!}x^3 + \dots + \frac{(-1)^{n+1}n!}{n!}x^n + \dots$$

$$= 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$$

Math 13 — Chapter 10.8 — Taylor and Maclaurin Series — In-class

2. Maclaurin series for $\sin x$ $(a = 0)$

4. Maclaurin series for $\ln(1 + x)$ $(a = 0)$

p 630 # 19

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

Thus $e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots + \frac{(-x)^n}{n!}$

$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + \frac{x^{2n}}{(2n)!} - \frac{x^{2n+1}}{(2n+1)!} + \dots$

Therefore $e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots + \frac{2x^{2n}}{(2n)!} + \dots$

So

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \dots$$