INFINITE SERIES (10.2)

As mentioned before , the name <u>series</u> refers to a (infinite) <u>sum</u> of elements of a sequence. Some books use <u>series</u> to refer to a <u>sequence</u> of (partial) sums (of a related series).

EX 1 Suppose we are given the sequence $\left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$

Form a new sequence whose elements are sums of the first K elements of the original sequence.

$$s_1 = a_1 = 1$$
$$s_2 = a_1 + a_2 = 1 + \tfrac{1}{2} = 3/2$$
$$s_3 = a_1 + a_2 + a_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} = 7/4$$
$$s_n = \sum_{i=1}^{n} a_i$$

This new sequence of (partial) sums, $\{s_n\}$, is called an <u>infinite series</u>.

We usually denote $\{s_n\}$ by $\sum_{n=1}^{\infty} a_n$.

The element $s_n = \sum_{i=1}^{n} a_i$ is called the <u>n^{th} partial sum</u>.

The element a_n is called the <u>n^{th} term</u>.

If there exists a (finite) number L s.t.

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

the L is the <u>sum</u> of the <u>series</u> and the <u>series converges</u>.

We often write
$$\sum_{n=1}^{\infty} a_n = L \quad .$$

If no limit exists, we say the series <u>diverges</u>.

FINDING THE SUM/LIMIT

One standard way to find the sum/limit is shown here:

Suppose $a_i = \frac{3}{10^i}$, i.e. $a_1 = \frac{3}{10}$, $a_2 = \frac{3}{10^2}$, $a_3 = \frac{3}{1000}$

Therefore $S_1 = a_1 = 3/10$

$S_2 = a_1 + a_2 = .33$

$S_3 = a_1 + a_2 + a_3 = .333$

Each a_i differs from the next one by a factor of $\frac{1}{10}$.

Therefore write out S_n and $\frac{1}{10} S_n$ and subtract.

$$S_n = 3/10 + 3/100 + 3/10^3 + \dots + 3/10^{n-1} + 3/10^n$$
$$\frac{1}{10} S_n = \qquad 3/100 + 3/10^3 + \dots + 3/10^{n-1} + 3/10^n + 3/10^{n+1}$$

$$\underbrace{S_n - \frac{1}{10} S_n}_{} = 3/10 \qquad\qquad - 3/10^{n+1}$$

$$\frac{9}{10} S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$\Rightarrow \quad S_n = \frac{3/10}{9/10}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

as $n \to \infty$; $\frac{1}{10^n} \to 0$, so $\lim_{n\to\infty} S_n = \frac{1}{3}$.

GEOMETRIC SERIES

The sum $\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{1}{3} = .333\ldots$ is a special kind of series called a <u>geometric series</u>.

<u>Def</u> A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

is called a geometric series with ratio <u>r</u>.

<u>EG</u> In $\sum_{n=1}^{\infty} 3/10^n$, $a = \frac{3}{10}$ and $r = \frac{1}{10}$

Using the same procedure as before, we can obtain the formula

$$s_n = \frac{a(1-r^n)}{1-r}.$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$.
Therefore. $\lim_{n\to\infty} s_n = \frac{a}{1-r}$ if $|r| < 1$.

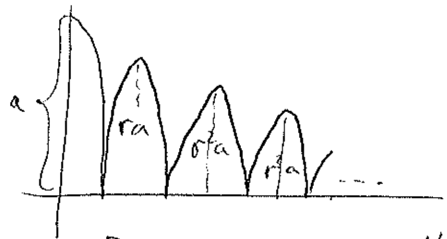
<u>Thm 1</u> (Geometric Series) p 586

If $|r| < 1$, the geometric series

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

CONVERGES to $\frac{a}{1-r}$. If $|r| \geq 1$, the series DIVERGES unless $a = 0$. (If $a=0$, $s_n = 0$ $\forall n$.)

p 587 EX 3 (Bouncing ball). A ball is dropped from a height of $\underline{a}$ meters. If it falls from a height h, it rebounds a distance rh. What is the TOTAL distance it travels (up + down).



$$s = a + (\overset{U}{ar} + \overset{D}{ar}) + (\overset{U}{ar^2} + \overset{D}{ar^2}) + \dots = a + 2ar + 2ar^2 + 2ar^3 + \dots$$

$$= a + \sum_{n=1}^{\infty} 2ar^n = a + \frac{2ar}{1-r} = \frac{a - ar + 2ar}{1-r}$$

$$= \frac{a(1+r)}{1-r}.$$

OTHER CLOSED FORMULAS

$\longrightarrow$ USE PARTIAL FRACTIONS

p587 (4 EX 8) By partial fractions, $\frac{1}{n(n+1)} = \frac{1}{n} + \frac{(-1)}{n+1}$

$\therefore$ we can use the second expression to see if

$$\sum_{1}^{\infty} \frac{1}{n(n+1)} \text{ converges or not.} \quad = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \quad = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$$

$$\sum_{1}^{\infty} \frac{1}{n(n+1)} = \sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots$$

It is obvious that $S_K = \sum_{1}^{K} \frac{1}{n(n+1)} = 1 - \frac{1}{K+1}$

Thus $\sum_{1}^{\infty} \frac{1}{n(n+1)} = 1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$

DIVERGENCE

Post-Thm 7 p588 (The n^{th} term test)

If $\lim_{n \to \infty} a_n \neq 0$ (the terms themselves not the sum),

or if $\lim_{n \to \infty} a_n$ fails to exist, then $\sum_1^{\infty} a_n$ diverges.

EX

$\sum_{n=1}^{\infty} n^2$ diverges since $n^2 \to \infty$

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges since $\frac{n+1}{n} \to 1 \neq 0$

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since $\lim_{n \to \infty} (-1)^{n+1}$ doesn't exist.

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If a series converges, then the terms themselves must converge to 0.

NOTE; the reverse is NOT true necessarily.

COMMENTS

$$a_n \not\to 0 \Rightarrow \sum^{\infty} a_n = \infty \quad \left[\sum \frac{n-1}{n}\right]$$

$$a_n \to 0 \Rightarrow \sum^{\infty} a_n = \infty \quad \left[\sum \frac{1}{n}\right]$$

or

$$\Rightarrow \sum^{\infty} a_n = c < \infty \quad \left[\sum \frac{1}{2^n}\right]$$

cf Improper Integrals (8.8)

Ex 3 p 507

$$\int_1^\infty \frac{dx}{x} \to \text{diverges} \quad \left(\text{similar to } \sum \frac{1}{n}\right)$$

$$\int_1^\infty \frac{dx}{x^2} \to \text{converges} \quad \left(\text{similar to } \sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\right)$$

COMBINING SERIES

<u>Thm 8 p 589</u> If $A = \sum_1^\infty a_n$ and $B = \sum_1^\infty b_n$ both exist and are finite, then

i) $\sum_1^\infty (a_n \pm b_n) = A \pm B$

ii) $\sum_1^\infty K a_n = K \sum_1^\infty a_n = KA$ (K any number).

NOTE; by letting $K = -1$ in ii) and using i) we also have $\sum_1^\infty (a_n - b_n) = A - B$.

<u>Corol</u> If $\sum_1^\infty a_n$ diverges, and if c is any number $\neq 0$, then $\sum_1^\infty c a_n$ also diverges.

<u>NOTE:</u> The multiplication and division versions of the above addition/subtraction rules do NOT hold. cf probs #83,84,85 p. 592

COMMENTS (p 590)

1. The addition or removal of a FINITE number of terms from a series has no effect on whether the series converges or diverges. If it CONVERGES, however, the extra (lost) terms WILL change the sum!

2. The indexing is merely a human convenience and does not change the convergence/divergence of a series. I.e.

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}$$

and

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^{n-1} - 2^{n-1}}{6^{n-1}}$$