SEQUENCES AND SERIES

Sequences refer to a (infinite) list of numbers. Usually, there is a formula or a method to describe the elements of a sequence.

E.G.

① $2, 4, 6, 8, 10, 12, \ldots$

$\Rightarrow$ even numbers – formula: $x_n = 2n$

② $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

$\Rightarrow$ prime numbers – no formula exists, but we can describe the elements.

③ $1\ 1\ 2\ 3\ 5\ 8\ 13\ 21 \ldots$

$\Rightarrow$ fibonacci numbers – $f_n = f_{n-1} + f_{n-2}$

Series refer to a (infinite) sum of elements of a sequence.

CONVERGING SEQUENCES

Let us start with several sequences:

I $0, 1, 2, 3, 4, \ldots, n-1$

II $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}$

III $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1-\frac{1}{n}$

IV $3, 3, 3, 3, \ldots, 3$

The 1st sequence is distinctively different from the other 3 in that there is no finite number that the sequence eventually gets "close to". The other 3 sequences eventually get very near to 0, 1, and 3 respectively.

If a sequence eventually gets close to one finite number, the number is called the limit of the sequence, and it is said to converge. More formally,

Bard p574

Def

The sequence $\{a_n\}$ converges to the number L (called its limit) iff $\forall\, \epsilon > 0$, $\exists\, N$, an integer > 0, s.t.

$$|a_n - L| < \epsilon \text{ whenever } n > N.$$

In other words, if you choose an ϵ, you can also find an N s.t. far enough out in the sequence, i.e. a_n s.t. $n > N$, a_n is ϵ-close to L. We usually write $\lim_{n\to\infty} a_n \Rightarrow L$

If a sequence does not have a limit, it is said to diverge.

EXAMPLES

I $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots (-1)^{n+1}(\frac{1}{n})$

converges to 0

II $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots (-1)^{n+1}(1-\frac{1}{n})$

diverges

(since as the sequence progress, it gets close to 2 numbers!, 1 and −1).

III $2, 2, 2, \ldots, 2$

converges to 2

diverge to $\pm\infty$

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COMBINING CONVERGENT SEQUENCES (10.1)

Thm 1, p 576

Let $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$. Assume both limits exist (and are finite!).

1) $\lim \{a_n + b_n\} = A + B$

2) $\lim \{K a_n\} = KA$ (where K is any number).

3) $\lim \{a_n \cdot b_n\} = A \cdot B$

4) $\lim \left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$ provided $B \neq 0$ and b_n is also never 0.

EG EX 3

① $\lim_{n\to\infty} \left(-\frac{1}{n}\right) = (-1) \lim \left(\frac{1}{n}\right) = (-1)\cdot 0 = 0$

② $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = \lim 1 - \lim \frac{1}{n} = 1 - 0 = 1$

Corol If $\{a_n\}$ diverges, and c is any number different from 0, then $\{c a_n\}$ diverges.

Sequence Sandwich Thm (of 2-2) (Thm 2 p576)

If $a_n \le b_n \le c_n \ \forall n > N$ (for some N) and
if $\lim a_n = \lim c_n = L$,
THEN $\lim b_n = L$ also.

EG

① Suppose $c_n \to 0$ and $|b_n| \le c_n$. Let $a_n = -c_n$.
Then $a_n = -c_n \le b_n \le c_n$ and $a_n \to 0$ and $c_n \to 0$
$\therefore b_n \to 0$.

② $\dfrac{\cos n}{n} \to 0$ since $|\cos n| \le 1$

and thus $0 \le \left|\dfrac{\cos n}{n}\right| = \dfrac{|\cos n|}{n} \le \dfrac{1}{n}$

and $\left\{\dfrac{1}{n}\right\} \to 0$.

This theorem is so named because b_n is sandwiched between a_n and c_n.

EVALUATING SEQ.

p577 **Thm 3** If $\lim_{n\to\infty} a_n = L$ and if f is a function that is continuous at L and defined at all the a_n's, then $\lim_{n\to\infty} f(a_n) \Rightarrow f(L)$.

E.G. (EX 5) Find $\lim_{n\to\infty} \sqrt{\frac{n+1}{n}}$.

Choose $f(x) = \sqrt{x}$ and $a_n = \frac{n+1}{n}$

$f(x)$ is continuous and $a_n \to 1$

$\therefore \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = f\left(\lim_{n\to\infty} \frac{n+1}{n}\right) = f(1) = \sqrt{1} = 1$

Thm 4: Let $f(x)$ be defined for all $x \geq n_0$ (i.e. continuous funct on real x) and $\{a_n\}$ is a sequence s.t. $a_n = f(n)$ (i.e. discrete funct on integer n) when $n \geq n_0$. If $\lim_{x\to\infty} f(x) = L$ (continuous for all real x) then $\lim_{n\to\infty} a_n = L$ also. (discrete, n only integers)

E.G. (EX 7) Find $\lim_{n\to\infty} \frac{\ln n}{n}$.

Look at the related function $\frac{\ln x}{x}$ and find its limit via L'Hôpital's rule.

$$\lim \frac{\ln x}{x} = \lim \frac{1/x}{1} = \lim \frac{1}{x} = 0$$

$\therefore \lim_{n\to\infty} \frac{\ln n}{n} = 0$ also.

COMMON LIMITS (Thm 5 p578)

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$ if $x > 0$

4. $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x)

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

EXAMPLES

1. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n =_{(5)} e^1 = e$

2. $\lim_{n\to\infty} \frac{100^n}{n!} =_{(6)} 0$

3. $\lim_{n\to\infty} \sqrt[n]{2n} = \lim_{n\to\infty} \sqrt[n]{2}\,\sqrt[n]{n} =_{(3)(2)} 1\cdot 1 = 1$

4. $\lim_{n\to\infty} \frac{\log n}{n^c} =_{l'H} \lim \frac{1/n}{c n^{c-1}} = \lim \frac{1}{c n^c}$

$= 0$ (similar to (1))

SAMPLE PROOFS

SKIP mp 18-5

1. $\lim\limits_{n\to\infty} \frac{\ln n}{n}$

By L'Hopital's rule we see

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$$

Since $\frac{\ln x}{x} = \frac{\ln n}{n}$ at integer points $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ also.

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5. $\lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

App 4,5
PJ AP-22

Let $a_n = \left(1+\frac{x}{n}\right)^n$

then $\ln a_n = n \ln\left(1+\frac{x}{n}\right)$

$$\lim_{n\to\infty} \ln a_n = \lim n \ln\left(1+\frac{x}{n}\right) = \lim \frac{\ln\left(1+\frac{x}{n}\right)}{1/n}$$

$$\overset{l'H}{=} \lim \frac{\frac{1}{1+\frac{x}{n}} \cdot \frac{-x}{n^2}}{-1/n^2} = \lim \frac{x}{1+x/n} = x$$

Since $\lim \ln a_n = x$

then $\lim e^{\ln a_n} = e^x = \lim a_n$

by Thm 3 p 577