LAGRANGE MULTIPLIERS (14.8) p 857

Look at this problem.

cf 14.7 prob 59

We have $12 \, m^2$ of cardboard.
We want to maximize volume of a box made from this.
The material is used only for sides and bottom – not top.
What are the dimensions?

Thus we want to maximize $V = x \cdot y \cdot z$ (volume)
subject to condition $2xz + 2yz + xy = 12$ (area of cardboard).

Solving the condition for z, we get $z = \dfrac{12 - xy}{2(x+y)}$, so volume can be rewritten as $V = xy\left(\dfrac{12-xy}{2(x+y)}\right) = \dfrac{12xy - x^2y^2}{2(x+y)}$

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \quad \text{and} \quad \frac{\partial V}{\partial y} \text{ is similar.}$$

Setting equal to 0, we get $y = 0, x = 0$ which makes no sense for a max vol.
or $12 - 2xy - x^2 = 0 = 12 - 2xy - y^2$

These imply $x^2 = y^2$ and $x = y$.

Using either of the last equations, we get
$$12 - 2x^2 - x^2 = 0 = 12 - 3x^2 \Rightarrow x^2 = 4 \Rightarrow \underline{x = 2}$$
Thus $\underline{y = 2}$ and $\underline{z = 1}$ $\therefore$ max vol is $2 \cdot 2 \cdot 1 = \underline{4}$

pg 860 An alternative approach uses <u>Lagrange Multipliers</u> which is based on the theory of gradients.
Remember that a gradient vector is perpendicular to a level curve of a function. If 2 level curves touch at a point, their gradients at the point of tangency must be parallel and thus one is a scalar multiple of the other.

Let one curve be the constraint level curve g. This is unique. Let the other curve be, in effect, a family of level curves based on values of input variables to f. We choose input values to produce the level curve that "touches" the constraint curve, i.e. whose gradients are parallel. If the gradients of f and g are parallel at (x_0, y_0, z_0) we have

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

and λ is called the Lagrange multiplier.

In this equation, $f(x,y,z)$ is what we want to minimize/maximize. It is an expression.

On the other hand, $g(x,y,z)=0$ is <u>an equation</u>, which represents the "constraint."

This approach is formalized in Thm 12 p 860

RE-DONE INITIAL EXAMPLE

$$V = x \cdot y \cdot z \qquad 2xz + 2yz + xy - 12 = g(x,y,z) = 0$$

Using Lagrange multipliers, we have

$$\nabla V = \lambda \nabla g$$

or, by components;

$$\text{①} \quad V_x = yz = \lambda g_x = \lambda(2z + y)$$

$$\text{②} \quad V_y = xz = \lambda g_y = \lambda(2z + x)$$

$$\text{③} \quad V_z = xy = \lambda g_z = \lambda(2x + 2y)$$

We may need ingenuity to solve this system of 3 equations!

$$x \cdot \text{①} \Rightarrow \quad xyz = \lambda(2xz + xy)$$

$$y \cdot \text{②} \Rightarrow \quad xyz = \lambda(2yz + xy)$$

$$z \cdot \text{③} \Rightarrow \quad xyz = \lambda(2xz + 2yz)$$

Top 2 equations imply

$$2xz + xy = 2yz + xy \Rightarrow xz = yz \Rightarrow \underline{x = y} \quad (\text{since } z \neq 0)$$

Last 2 equations imply

$$2yz + xy = 2xz + 2yz \Rightarrow xy = 2xz \Rightarrow y = 2z$$

Using these values in $g(x,y,z)$ to replace x and y, we get

$$2^2z^2 + 2^2z^2 + 2^2z^2 = 12 \Rightarrow 12z^2 = 12 \Rightarrow z^2 = 1 \Rightarrow z = 1$$

$\therefore\ x = 2,\ y = 2$ as before

ANOTHER DETAILED EXAMPLE

EX 2
p 858

Find the points on the surface of the hyperbolic cylinder

$$x^2 - z^2 - 1 = 0$$

closest to the origin.

1st Try

The distance from a point (x, y, z) to the origin is

$$d = \sqrt{x^2+y^2+z^2}$$

Let $f(x,y,z) = d^2 = x^2+y^2+z^2$.

Minimizing f also minimizes d (and is easier to work with)

Thus we want to minimize $f(x,y,z)$ subject to $x^2-z^2-1=0$

We re-write the eq of the surface (i.e. The constraint) to get

$$z^2 = x^2 - 1$$

and plug it into f to get a function of 2 var's, $h(x,y)$

$$h(x,y) = x^2+y^2+(x^2-1) = 2x^2+y^2-1$$

Taking the derivatives of h and setting equal to 0, we get

$$h_x = 4x = 0 \quad \text{and} \quad h_y = 2y = 0$$

$$\Rightarrow \quad x = y = 0$$

But there are no points on surface where $x = y = 0$!

$\Rightarrow$ If we re-wrote the equation of the surface differently, i.e. $x^2 = z^2 + 1$, we would get the correct answer.

2nd Try Using Lagrange multipliers.

Imagine a sphere centered at the origin, i.e. $x^2+y^2+z^2=a^2$, which touches the surface of the cylinder.

At each pt of contact, the surface and sphere have the same tangent plane and normal line.

Let us represent the sphere and the surface of the cylinder as level curves of the functions.

$$f(x,y,z) = x^2+y^2+z^2-a^2=0 \quad \text{and}$$
$$g(x,y,z) = x^2-z^2-1 = 0$$

Then the gradients ∇f and ∇g will be parallel where the surfaces touch. Thus, we should be able to find λ s.t.

$$\nabla f = \lambda \nabla g$$

$$\therefore \quad \langle 2x, 2y, 2z \rangle = \lambda \langle 2x, -2z \rangle$$

$$\Rightarrow \quad \begin{aligned} 2x &= 2\lambda x \\ 2y &= 0 \\ 2z &= -2\lambda z \end{aligned}$$

We want a λ that gives a point on $x^2-z^2-1=0 \Rightarrow x^2=z^2+1$

This implies $x^2 \geq 1$ i.e. $x \neq 0$

$$\therefore \quad 2x = 2\lambda x \Rightarrow 2 = 2\lambda \Rightarrow \lambda = 1$$

$$\Rightarrow \quad 2z = -2z \Rightarrow z = 0$$

$\therefore$ points must have the form $(x, 0, 0)$

But $(x,0,0)$ must also be on $x^2=z^2+1 \therefore x^2 = 0+1=1$

$\Rightarrow x = \pm 1 \qquad \therefore$ points are $(\pm 1, 0, 0)$

TWO CONSTRAINTS

If we need to find the max/min values of $f(x,y,z)$ subject to 2 constraints g and h, we use 2 Lagrange multipliers with the gradients

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

EX Find the max of $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ $(\sim g(x,y,z))$ and cylinder $x^2 + y^2 = 1$ $(\sim h(x,y,z))$

cf problems 37-44

Equating the components of the gradients we get

$$1 = \lambda + 2x\mu$$
$$2 = -\lambda + 2y\mu$$
$$3 = \lambda + 0$$

$$-2 = 2x\mu \Rightarrow x = -\frac{1}{\mu}$$

$$5 = 2y\mu \Rightarrow y = \frac{5}{2\mu}$$

Substituting into $h(x,y,z)$, we get.

$$\left(\frac{-1}{\mu}\right)^2 + \left(\frac{5}{2\mu}\right)^2 = 1 = \frac{1}{\mu^2} + \frac{25}{4\mu^2} = \frac{4+25}{4\mu^2} = \frac{29}{4\mu^2}$$

$$\Rightarrow 4\mu^2 = 29 \Rightarrow \mu = \pm\frac{\sqrt{29}}{2}$$

$$\Rightarrow x = \mp\frac{2}{\sqrt{29}}, \quad y = \pm\frac{5}{\sqrt{29}}, \quad z = 1 - x + y = 1 \pm \frac{7}{\sqrt{29}}$$

$$\therefore f(x,y,z) = x + 2y + 3z = \frac{\mp 2}{\sqrt{29}} \pm \frac{10}{\sqrt{29}} + 3 \pm \frac{21}{\sqrt{29}} = \pm\frac{29}{\sqrt{29}} + 3$$

$\therefore$ max is $3 + \sqrt{29}$