MAX AND MIN (AGAIN) : (14.7) p. 848

Back in Chpt , we looked at optimization in 2D. Now we try the same techniques for 3D.
3D definition are parallel to 2D definition.

p848 **Defs** A function $z = f(x,y)$ has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b) (i.e. for all points (x,y) in a disk centered at (a,b)).
The number $f(a,b)$ is called the local maximum value.
Similar definition for local minimum.

p852 **Def** If the inequality hold for all points in the domain of $f(x,y)$, then f has an absolute maximum (minimum).

p849 **Thm**[10] If f has a local max/min at (a,b) and the first order partial derivatives of f exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$, i.e. $\nabla f \big|_{(a,b)} = \vec{0}$

p849 **Def** A point (a,b) is called a critical point (a stationary point) of f if $f_x(a,b) = 0 = f_y(a,b)$ (or one partial doesn't exist).

Note: If f has a local max/min at (a,b), then (a,b) is a critical point of f.
But, not every critical point is a max/min point!!

Given $z = f(x,y)$ and given (a,b) is a critical point. If there are points close to (a,b) s.t. $f(x,y) > f(a,b)$ for some (x,y) and $f(x,y) < f(a,b)$ for other (x,y), then (a,b) is called a <u>saddle point</u>.

In other words, (a,b) is a max point along certain curves and a min point along other curves.

SECOND DERIVATIVE TEST (p 850)

In 2D, there is only one 2nd derivative, but in 3D there are 4 possible 2nd derivatives, although for many cases $f_{xy} = f_{yx}$. All 4 of the 2nd derivatives are used to obtain a value used in the 3D version of the 2nd Derivative Test.

Thm 11

2nd Deriv Test Given $f(x,y)$ with continuous 2nd partials on a disk centered at (a,b). Suppose $f_x(a,b) = 0 = f_y(a,b)$, resulting in (a,b) being a critical point.
Compute the discriminant D of f at (a,b):

$$D = D(a,b) = \left.\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}\right|_{(a,b)} = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Then

a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum
b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
c) If $D < 0$, then $f(a,b)$ is a saddle point.
d) If $D = 0$, the test is inconclusive & we need other tests to determine what we have.

EXAMPLES

cf p.855 # 20

also p 850-52 EX 1-5

Find local max/min & saddle points of
$$f(x,y) = x^4 + y^4 - 4xy + 1$$

① Find critical points:

Take partials: $f_x = 4x^3 - 4y$ $\qquad f_y = 4y^3 - 4x$

Set to 0: $4x^3 - 4y = 0$ $\qquad 4y^3 - 4x = 0$

Simplify: $x^3 - y = 0$ $\qquad y^3 - x = 0$ $\qquad y = x^3$

Subst. one eq in the other: $0 = (x^3)^3 - x = x^9 - x = x(x^8 - 1)$

$= x(x^4-1)(x^4+1) = x(x^2-1)(x^2+1)(x^4+1)$

$= x(x-1)(x+1)(x^2+1)(x^4+1)$

$\Rightarrow x = 0, 1, -1 \qquad \Rightarrow y = 0, 1, -1$

$\Rightarrow$ critical points are $(0,0), (1,1), (-1,-1)$

② Compute Discriminant:

Take 2nd partials: $f_{xx} = 12x^2 \qquad f_{yy} = 12y^2$

$f_{xy} = -4 \qquad f_{yx} = -4$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - (-4)^2$

$= 144x^2y^2 - 16$

③ Evaluate $D(x,y)$ at each critical point.

$D(0,0) = 0 - 16 = -16 < 0 \quad \Rightarrow \quad f(0,0) = 1$ is a saddle point.

$D(1,1) = 144 - 16 = 128 > 0$, $f_{xx}(1,1) = 12 \cdot 1 > 0 \Rightarrow f(1,1) = -1$ is a local minimum.

$D(-1,-1) = 144 - 16 = 128 > 0$, $f_{xx}(-1,-1) = 12 \cdot 1 > 0 \Rightarrow f(-1,-1) = -1$ is also a local minimum.

#2 Find the shortest distance from $(1,0,-2)$ to plane $x+2y+z=4$ cf p 856 #56

→ doesn't initially seem related to max-min on surfaces.

Distance from any point (x,y,z) to $(1,0,-2)$ is $d=\sqrt{(x-1)^2+y^2+(z+2)^2}$

If (x,y,z) is on plane, then $z=4-x-2y$, thus the distance is $d=\sqrt{(x-1)^2+y^2+(6-x-2y)^2}$

Let $f(x,y)=d^2=(x-1)^2+y^2+(6-x-2y)^2$ ← Surface 3D.

Minimizing f also minimizes d.

ct.

#3 OPTIMIZATION PROBLEM

cf EX 7, p 854

Rectangular box cannot have sum of length + girth larger than 108 in. (ie. $x+2y+2z=108$ max). What are dimensions of box of largest volume?

$V=xyz \qquad x=108-2y-2z$

$\Rightarrow V(y,z)=(108-2y-2z)yz=108yz-2y^2z-2yz^2$

ct.

ABSOLUTE MAX/MIN VALUES (p 852)

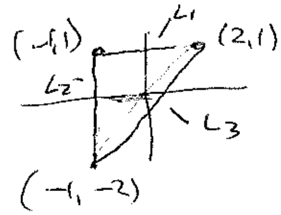
As with the 2D case (cf 4.1, p 233), we can also determine the absolute max/min on a closed, bounded set (i.e., a finite set that includes the boundary points).

To find the absolute max/min, we find local max/min and check against values on boundary.

cf. p 855 #32 also Ex 6 p 853

EX. Find abs. max/min for $f(x,y) = x^2 + 2xy + 3y^2$ on D, where D is the triangle with vertices $(-1,1)$, $(2,1)$, and $(-1,-2)$.

Prelim: Let L_1 be line between $(-1,1)$ and $(2,1)$
L_2 between $(-1,1)$ and $(-1,-2)$
L_3 between $(2,1)$ and $(-1,-2)$



$\therefore$ L_1: $y = 1$, L_2: $x = -1$, L_3: $y = x - 1$

Now $f_x = 2x + 2y$ and $f_y = 2x + 6y$

$$\left.\begin{array}{l} 2x + 2y = 0 \\ 2x + 6y = 0 \end{array}\right\} \Rightarrow x = y = 0 \quad \Rightarrow \text{critical point at } (0,0) \text{ and } f(0,0) = 0$$

Along L_1 we have $f(x,1) = x^2 + 2x + 3 \quad (-1 \le x \le 2)$

Thus $\frac{df}{dx} = 2x + 2 = 0 \Rightarrow x = -1 \quad \frac{d^2f}{dx^2} = 2 > 0 \therefore$ min at $x = -1$

$\Rightarrow f(-1,1) = 2$. at (other) endpoint $f(2,1) = 11$.

Along L_2 we have $f(-1,y) = 1 - 2y + 3y^2 \quad (-2 \le y \le 1)$

Thus $\frac{df}{dy} = -2 + 6y = 0 \Rightarrow y = \frac{1}{3} \quad \frac{d^2f}{dy^2} = 6 > 0 \therefore$ min at $y = \frac{1}{3}$

$\Rightarrow f(-1, \frac{1}{3}) = \frac{2}{3}$

Along L_3 we have $f(x, x-1) = x^2 + 2x(x-1) + 3(x-1)^2 = 6x^2 - 8x + 3 \quad (-1 \le x \le 2)$

Thus $\frac{df}{dx} = 12x - 8 = 0 \Rightarrow x = \frac{2}{3} \quad \frac{d^2f}{dx^2} = 12 > 0 \therefore$ min at $x = 2/3$

$\Rightarrow f(\frac{2}{3}, -\frac{1}{3}) = \frac{1}{3}$ at endpt $f(-1,-2) = 17$

Putting all this together.

along L_1, min $f(-1,1)=2$ max $f(2,1)=11$

L_2, min $f(-1,\frac{1}{3})=\frac{2}{3}$ max $f(-1,-2)=17$

L_3, min $f(\frac{2}{3},\frac{-1}{3})=\frac{1}{3}$ max $f(-1,-2)=17$

$\left.\right\} \leftarrow$ max

min is within region at $f(0,0)=0$