CURVES IN 3D (OVERVIEW) (cf 13.1 p751)

A curve in 3D is usually described parametrically, similar to how a line is described. We think of a curve as a path made by the motion of a particle in space over a period of time. Thus, we have

$$x = f(t), \quad y = g(t), \quad z = h(t)$$ when t has a specific range of values

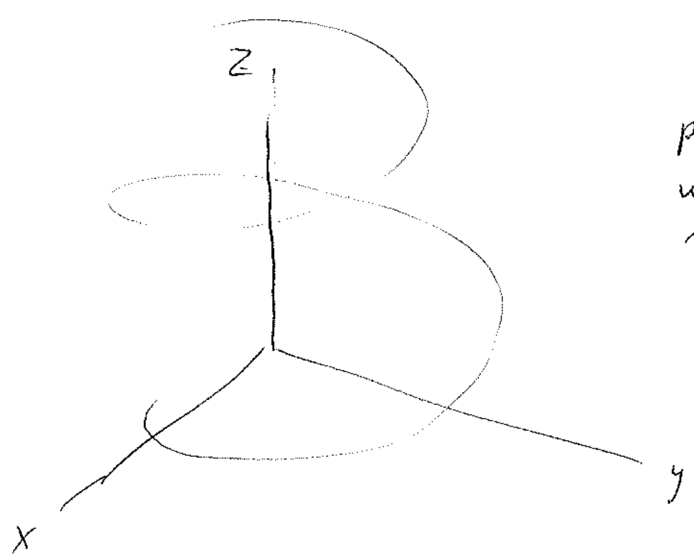
A curve can also be represented as a vector, e.g.

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

when the "tip" of the vector "points" to the particle moving in space and describing a curve.

A simple curve is a helix.

$$r(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$$



In the xy plane projection of this curve, we get a circle of radius = 1, since $\cos^2 t + \sin^2 t = 1$

TANGENT PLANES + NORMAL LINES (14.6) p839

Let $w = f(x,y,z)$ and its partial derivatives be continuous.

Let S be the level surface (cf 14.1, p796), $f(x,y,z)=c$.

Let C be a curve on S described by $\vec{r}(t) = (x=x(t), y=y(t), z=z(t))$

s.t. C passes through point $P_0(x_0,y_0,z_0)$ on S.

By the chain rule, differentiating $S: f(x,y,z)=c$, we get

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$$

But this can be re-written as $\nabla f \cdot \vec{r}\,'(t) = 0$ or

$$\nabla f \cdot \vec{v} = 0$$

where $\vec{v} = \vec{r}\,'(t)$, the velocity vector of the curve S.

at P_0, we have

$$\nabla f\Big|_{P_0} \cdot \vec{v}(t_0) = 0$$

If $\vec{r}$ is the position vector of C, then $\vec{v}$, the velocity vector is the <u>tangent vector</u> to C (cf 13.1, p755).

Since C, and thus $\vec{r}(t)$, are arbitrary curves passing through P_0, this says ∇f is a <u>Normal vector</u> to the surface $S: f(x,y,z)=c$, (i.e. $\perp$ to any tangent) at P_0

With a normal vector to the surface, we can construct the eq for a <u>tangent plane</u> (cf. 12.5, p 735)

We use $\nabla f|_{P_0}$ as the normal vector and $P_0(x_0, y_0, z_0)$ as the pt to obtain.

p 839 (1) $$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$$

The corresponding <u>normal line</u> is obtained using $\nabla f|_{P_0}$ as the parallel direction vector.

$$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)} = t$$

or

(2) $$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = t \Rightarrow x - x_0 = f_x\big|_{P_0} t \Rightarrow x = x_0 + f_x\big|_{P_0} t$$

etc. for y and z.

2-DIM SURFACES (p 840)

If a surface S is described by $z = f(x, y)$,
then $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$
is the tangent plane. (unless z is a constant in which case the last term disappears).

EXAMPLE

#1 Find (a) tangent plane and (b) Normal line to

$x^2+2y^2+3z^2=21$ at $(4,-1,1)$

cf EX1 p839

$\nabla F = \langle 2x, 4y, 6z \rangle$

$\nabla F\big|_{P_0} = \langle 8, -4, 6 \rangle$

Plane: $8(x-4)-4(y+1)+6(z-1)=0$

$= 8x-32-4y-4+6z-6$

$\Rightarrow 8x-4y+6z = 32+4+6 = 42$

$\Rightarrow \underline{4x-2y+3z=21}$

Line: $\dfrac{x-4}{8} = \dfrac{y+1}{-4} = \dfrac{z-1}{6}$

ESTIMATING CHANGE IN A DIRECTION

In 3.11 (p 201) we introduced differentials and saw how they could be used to estimate a change in the function value. I.e. given $y = f(x)$, we have

$$dy = df = f'(x)\,dx$$

If we know a specific value of x, eg, $x = a$, then

$$dy = f'(a)\,dx$$

can indicate how much y changes if we move away from $x = a$ in the x-direction

In 3D, we have

$$df = \left(\nabla f\big|_{P_0} \cdot \vec{u}\right) ds \qquad p\ 841$$

where $\nabla f\big|_{P_0} \cdot u$ is the directional derivative and ds is the small distance in the direction of $\vec{u}$

LINEAR APPROXIMATION (p 842)

In the 2D case, we saw that a tangent line was a good linear approximation to a curve NEAR the point of tangency. cf. 3.11 p 203

A similar fact holds in the 3D case.

Given a surface $z = f(x,y)$ and a point on the surface $(x_0, y_0, f(x_0,y_0))$, the linearization of $f(x,y)$ at (x_0,y_0) is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

EXAMPLE

p 846 #25

Find linearization of $f(x,y)$ at given pts

$f(x,y) = x^2+y^2+1$ at a) $(0,0)$, (b) $(1,1)$

cf EYS p 842

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\left[\begin{array}{l} f_x(x,y) = 2x \\ f_y(x,y) = 2y \end{array}\right.$$

$$L(x,y)\Big|_{(0,0)} = f(0,0) + 2(0)(x-0) + 2\cdot 0\,(y-0)$$

$$= 1$$

$$L(x,y)\Big|_{(1,1)} = (1+1+1) + 2(1)(x-1) + 2\cdot 1\,(y-1)$$

$$= 3 + 2x-2 + 2y-2$$

$$= 2x+2y-1$$

DIFFERENTIALS p 843

If we want to describe df in terms of dx and dy rather than ds in the direction of the direction vector $\vec{u}$, we use the (total) differential of $f(x,y)$

p 844

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

As in the 2D case, this can be used to approximate the change in f near a given point.

Note that the discrete change is

p 843 bottom

$$\Delta f = f(x_0+\Delta x,\ y_0+\Delta y) - f(x_0,y_0)$$

For small values of Δx and Δy, $\Delta f \approx df$.

EG. Given $z = f(x,y) = x^2 + 3xy - y^2$

① find dz

② compute dz and Δz as x moves $2 \to 2.05$ and y moves $3 \to 2.96$

① $$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \qquad \frac{\partial z}{\partial x} = 2x+3y \qquad \frac{\partial z}{\partial y} = 3x-2y$$

$$= (2x+3y)\,dx + (3x-2y)\,dy$$

② assuming $dx = \Delta x = .05$ and $dy = \Delta y = -.04$ (and $x=2$, $y=3$)

we have
$$dz = (2\cdot 2 + 3\cdot 3)\cdot .05 + (3\cdot 2 - 2\cdot 3)(-.04)$$
$$= 13\cdot .05 - 0 = .65$$

$$\Delta z = f(2.05, 2.96) - f(2,3)$$
$$= 2.05^2 + 3(2.05)(2.96) - 2.96^2 - (2^2 + 3\cdot 2\cdot 3 - 3^2)$$
$$= 4.2025 + 18.204 - 8.7616 - (13)$$
$$= \underline{.6449}$$

Note $dz \approx \Delta z$

This shows how much the surface height has changed when one shifts location from $(2,3)$ to $(2.05, 2.96)$.

FUNCTIONS OF THREE OR MORE VARIABLES

p 845

We can extend definitions of linear approximations, increments, differentials, etc to functions of more than 2 variables.

EG. a linear approximation of $w = f(x, y, z)$ is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

(The increment is

$$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$
)

The differential is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz.$$