INVERSE HYP FUNCTIONS (7.3

Given $x = \sinh y$

then the inverse function is $y = \sinh^{-1} x$

or, by convention, $y = \arg\sinh x$

With the circular trig functions, we have problems with the inverse functions since they can have more than 1 value. ∴ We must define "principal" values. Do we have similar problems with the hyperbolic trig functions? The problems here are not as serious.

- No prob with $\arg\sinh u$ since it is 1-1.
- For $y = \arg\cosh u$, u must be ≥ 1 + we take the principal value to be $y \geq 0$
- For $y = \arg\operatorname{sech} u$ we restrict u s.t. $0 < u \leq 1$ and take y to be > 0.
- no problems with $\arg\tanh u$, $\arg\coth u$, $\arg\operatorname{csch} u$.

REMEMBER:

$$y = \operatorname{csch}^{-1} x$$

$$\Longleftrightarrow y = \arg\operatorname{csch} x$$

$$\Longleftrightarrow x = \operatorname{csch} y$$

COMMENT: Since hyp. funct are defined in terms of e^u, it is possible to obtain a definition of the inverse function in terms of the inverse of e^u, i.e. $\log u$.

cf book p. 446 Box

eg. $y = \arg\tanh x = \frac{1}{2}\log\frac{1+x}{1-x}$ for $|x| < 1$

DERIVATIVES/INTEGRALS OF INV-HYP-TRIG-FUNCT.

Method is basically the same for all functions.

Look at $\frac{d}{dx}(\arg\sinh u)$.

Let $y = \arg\sinh u \iff u = \sinh y$

$$\frac{du}{dx} = \cosh y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}\frac{du}{dx}$$

But we want right side in terms of u.

$$\cosh^2 y - \sinh^2 y = 1$$

$$\therefore \cosh^2 y = 1 + \sinh^2 y = 1 + u^2$$

$$\therefore \cosh y = \sqrt{1+u^2} \quad \text{since } \cosh y \geq 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} = \frac{d}{dx}(\arg\sinh u) \quad //$$

From this we can get the corresponding integral form

$$\int \frac{du}{\sqrt{1+u^2}} = \arg\sinh u + C \quad //$$

OTHER RULES:

$$d(\cosh^{-1} u) = \frac{du}{\sqrt{u^2-1}} \quad u > 1$$

$$d(\tanh^{-1} u) = \frac{du}{1-u^2} \quad |u| < 1$$

$$d(\coth^{-1} u) = \frac{du}{1-u^2} \quad |u| > 1$$

$$d(\operatorname{sech}^{-1} u) = \frac{-du}{u\sqrt{1-u^2}} \quad 0 < u < 1$$

$$d(\operatorname{csch}^{-1} u) = \frac{-du}{|u|\sqrt{1+u^2}} \quad u \neq 0$$

EXAMPLE

$$\int_0^{\sqrt{5}} \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{4(1+(\frac{x}{2})^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1+(\frac{x}{2})^2}}$$

let $u = \frac{x}{2}$
$2du = dx$

$$= \frac{2}{2}\int \frac{du}{\sqrt{1+u^2}} = \arg\sinh u + C$$

$$= \arg\sinh\left(\frac{x}{2}\right) + C$$

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old sol.

$x = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4+4\tan^2\theta}} = \frac{2}{2}\int \frac{\sec^2\theta\, d\theta}{\sec\theta}$$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + C = \log\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

cf Box on p 446 for relations between inverse trig functions + log

Natural occurrence of hyp. functions.

in <u>catenary curve</u> — hanging chain

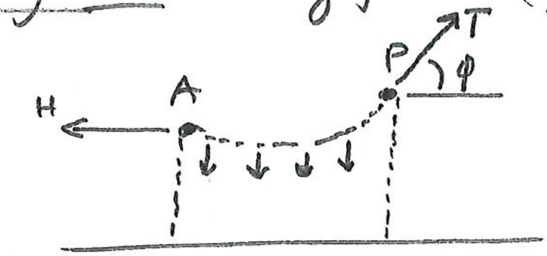
~~(6.10)~~ (9,3)

~~[13.8]~~ (5)

~~31-6~~ 31-2c

(581 7)

~~p477~~[8] → St. Louis arch. (READ ALOUD)



← ends touching ground i.e. equilibrium position.

Let T be tangential, (Tension) force vector at P

H be horizontal tension at A

$W = ws$ is the weight of s feet of cable at w-lbs/ft.

7,3
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Now horizontal vectors must be equal

$\therefore \quad H = T\cos\varphi$

also Vertical vectors must be equal:

$\therefore \quad W = ws = T\sin\varphi$

$\therefore \quad \dfrac{T\sin\varphi}{T\cos\varphi} = \tan\varphi = \dfrac{W}{H}$ But $\tan\varphi = \dfrac{dy}{dx}$ at pt. P.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{ws}{H}$ where s is the only possible variable

But s is length and $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{w}{H}\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

claim: $y = a\cosh\left(\frac{x}{a}\right)$ is sol. to this diff. eq.

$\dfrac{d}{dx}\left(a\cosh\frac{x}{a}\right) = a\left(\sinh\frac{x}{a}\right)\frac{1}{a} = \sinh\frac{x}{a}$ //

$\dfrac{d^2y}{dx^2} = \left(\cosh\frac{x}{a}\right)\left(\frac{1}{a}\right)$ //

check:

$\dfrac{1}{a}\cosh\dfrac{x}{a} \stackrel{?}{=} \dfrac{w}{H}\sqrt{1+\sinh^2\dfrac{x}{a}} = \dfrac{w}{H}\sqrt{\cosh^2\dfrac{x}{a}} = \dfrac{w}{H}\cosh\dfrac{x}{a}$

$\therefore$ True if $a = \dfrac{H}{w}$ i.e. <u>$y = a\cosh\frac{x}{a}$ is a sol to catenary diff eq.</u>