DIRECTIONAL DERIVATIVES (14.5) p830

Partial derivatives are special cases of the more general concept of "directional derivative." In particular, if $z=f(x,y)$, f_x is the rate of change in the x direction, that is, in the direction specified by the vector $\vec{i}$. Similarly with f_y.

It is useful to be able to compute a derivative in an arbitrary direction associated with a direction vector $\vec{u}$.

p831 **Def** The <u>directional derivative</u> of f at $P_0(x_0, y_0)$ in the direction of a <u>unit</u> vector $\vec{u} = a\vec{i} + b\vec{j}$ $(= \cos\theta\,\vec{i} + \sin\theta\,\vec{j})$ (where $\theta = \arctan\frac{b}{a}$) is.

$$D_u f(x_0, y_0) = \frac{df}{ds}\bigg|_{u, P_0} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

(if the limit exist) NOTE: h is measured along the direction vector u.

p832 Note if $\vec{u} = \vec{i}$, then $D_{\vec{i}} f = f_x$ and if $\vec{u} = \vec{j}$, then $D_{\vec{j}} f = f_y$.

Thm If f is a diff'ble function of x and y, then f has a directional derivative in the direction of any <u>unit</u> vector $\vec{u} = \langle a, b\rangle$ and

p833 line before eq (3)

$$\begin{aligned} D_u f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\ &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta. \end{aligned}$$

or $\displaystyle \frac{df}{ds} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$

E6

$f(x,y) = x^2y^3 + 2x^4y$

Find direct. deriv at $(1,-2)$ in direct $\theta = \frac{\pi}{3}$ $= \vec{\mu} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$f_x = 2xy^3 + 8x^3y$

$f_y = 3x^2y^2 + 2x^4$

$$D_\theta f = (2xy^3 + 8x^3y)\cos\frac{\pi}{3} + (3x^2y^2 + 2x^4)\sin\frac{\pi}{3}$$

$$\Big|_{(1,-2)} = (2\cdot 1\cdot(-8) + 8\cdot 1\cdot(-2))\,\frac{1}{2} + (3\cdot 1\cdot 4 + 2\cdot 1)\,\frac{\sqrt{3}}{2}$$

$$= (-16-16)\frac{1}{2} + (12+2)\sqrt{3}/2$$

$$= \underline{-16 + 7\sqrt{3}}$$

GRADIENT (p 833)

We can re write the directional derivative
$D_u f = f_x(x,y)\,a + f_y(x,y)\,b$ in the form of a
dot product of 2 vectors since it is a sum of products.
Thus

$$D_u f = \langle f_x, f_y \rangle \circ \langle a, b \rangle$$

The second vector is the direction vector $\vec{u}$ (unit vector)
The first vector, consisting of partial derivatives of the given
Middle 833 function f, is given the name "gradient of f"
and is designated grad f or ∇f (pronounced del f)

To summarize:

Def If f is a function of several variables, the gradient of f, ∇f
is the vector function defined by

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

or

p 836 $$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Thus the directional derivative can be written as

Thm 9 p 833

$$D_u f = \nabla f \circ \vec{u}$$

EX.

Given $f(x,y) = 5xy^2 - 4x^3y$, $P(1,2)$, $\vec{u} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

a) find gradient

b) evaluate gradient at P

c) find rate of change of f at P in direction of $\vec{u}$.

NOTE: $\vec{u}$ is a <u>unit vector</u>. If it was not, we would need to normalize it first.

a) $\nabla f = \langle 5y^2 - 12x^2y, 10xy - 4x^3 \rangle$

b) $\nabla f|_P = \langle 5\cdot 4 - 12\cdot 1\cdot 2, 10\cdot 1\cdot 2 - 4\cdot 1 \rangle$

$= \langle 20 - 24, 20 - 4 \rangle = \langle -4, 16 \rangle$

c) This "rate of change" is merely the directional derivative in the direction of $\vec{u}$.

I.e. $D_u f|_P = \nabla f|_P \cdot \vec{u}$

$= \langle -4, 16 \rangle \cdot \langle \frac{5}{13}, \frac{12}{13} \rangle$

$= -\frac{20}{13} + \frac{192}{13} = \underline{\underline{\frac{172}{13}}}$

<u>NOTE:</u> It is a common "trick" in certain areas of math to rewrite expressions (esp. sums of products) in terms of <u>vectors</u>. Thus it's important to continue to try to understand vectors in 3D.

USE OF GRADIENT AND DIR. DERIV (p 834)

The dot product of A and B can be defined as

$$A \cdot B = |A||B| \cos\theta$$ cf 12.3 p 718 Thm 1

where θ is angle between the 2 vectors

Therefore $D_u f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$

but since $\vec{u}$ is a unit vector, we have $D_u f = |\nabla f| \cos\theta$

$\cos\theta$ is maximum when $\theta = 0$ $\therefore \cos\theta = 1$

Thus, if ∇f and $\vec{u}$ point in the same direction $D_u f$ is at its maximum.

In other words, <u>the directional derivative $D_u f$ has its max. value in the direction of the gradient ∇f.</u>

Or, ∇f gives the direction of the greatest increase (<u>decrease</u> if neg.) of function f and $|\nabla f|$ gives the max value of $\frac{df}{ds}$ (for all possible direction).

A direction $\vec{u}$ orthogonal to ∇f is a direction of zero change in f.

EXAMPLES

#1 Find the maximum rate of change of $f(x,y) = xe^{-y} + 3y$ at $P(1,0)$ and the direction in which it occurs. (Max rate of change = max deriv.)

cf. EX3 p 834

Max rate of change is ∇f: $f_x = e^{-y} + 0$, $f_y = x(-1)e^{-y} + 3$

$$\therefore \nabla f = \langle e^{-y}, -xe^{-y}+3 \rangle$$

$$\therefore \nabla f \Big|_{(1,0)} = \langle e^0, -1e^0+3 \rangle = \underline{\underline{\langle 1, 2 \rangle}}$$

$\therefore$ max slope is in direction of $\langle 1, 2 \rangle \parallel \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \vec{u}$.

Max rate of change is $D_u f \Big|_P = \nabla f \cdot u = \langle 1, 2 \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$$= \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \underline{\underline{\sqrt{5}}}$$

or $|\langle 1, 2 \rangle| = \sqrt{1^2+2^2} = \underline{\underline{\sqrt{5}}}$

#2 Suppose you are climbing a hill whose shape is given by

$$f(x,y) = 1000 - 0.01x^2 - 0.02y^2$$

and you are standing at $P_0\ (60, 100, 764)$

a) In which direction should you proceed to reach the top of the hill the fastest?

b) If you climb in that direction, what is your angle of ascent (above the horizontal)?

a) $\nabla f = \langle f_x, f_y \rangle = \langle -.02x, -.04y \rangle$

$\therefore \nabla f_{P_0} = \langle -.02 \cdot 60, -.04 \cdot 100 \rangle = \langle -1.2, -4 \rangle$

or $-1.2\vec{i} - 4\vec{j}$

b) $D_u f = |\nabla f| = |\langle -1.2, -4 \rangle| = \sqrt{1.2^2 + 4^2} = \sqrt{1.44 + 16}$

$= \sqrt{17.44}$

But $D_u f$ is slope. $\therefore$ $D_u f$ = tangent of angle of inclination

$\therefore$ $\theta = \arctan \sqrt{17.44} \approx 76.5°$

(skip rest of sect.)