PARTIAL DERIVATIVES AND THE CHAIN RULE (14.4)

p 821

In the 2D case, if $w = f(x)$ and $x = g(t)$ and one want $\frac{dw}{dt}$, one uses the chain rule (cf. 3.6, p 164)

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

We now extend this to 3D.

FUNCTIONS OF TWO VARIABLES

p822

Thm 5 If $w = f(x, y)$ is differentiable, and $x = x(t)$, and $y = y(t)$ are both diff'ble functions of t, then f is a diff'ble function of t and we have the following version of the chain rule in 3D.

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Note that $\frac{dw}{dt}$ is a "regular" derivative, since, via $x(t)$ and $y(t)$ w can be rewritten as a function of a single variable

EX Given $z = x^2 y + x y^2$
$x = 2 + t^4 \quad y = 1 - t^3$

$$\Rightarrow \frac{dx}{dt} = 4t^3 \qquad \frac{dy}{dt} = -3t^2$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy + y^2)\, 4t^3 + (x^2 + 2xy)(-3t^2)$$

[One could re-write this in terms of $\underline{t}$.

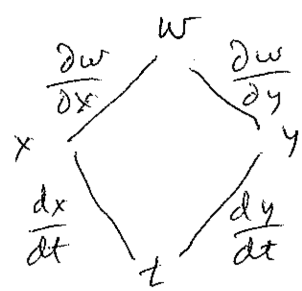
$$= \left(2(2+t^4)(1-t^3) + (1-t^3)^2\right) 4t^3$$
$$+ \left((2+t^4)^2 + 2(2+t^4)(1-t^3)\right)(-3t^2)$$

Sometimes, this is not done for simplicity.

BRANCH DIAGRAMS (MEMORY AIDS)

As a memory aid, some books use diamond (or tree) diagrams to assist in remembering the chain rule. For example, given $w = f(x, y)$, $x = x(t)$, $y = y(t)$, we draw a diamond and associate each node with a variable and each link with a derivative.

Thus,



Each path from the top node, w, to the bottom node, t, represents a <u>product</u> of derivatives and the chain rule consists of the <u>sum</u> of all such paths/products.

There are <u>2</u> paths from w to t: ① $w \to x \to t$ and ② $w \to y \to t$.

The corresponding products of derivatives are:

for ① $\frac{\partial w}{\partial x} \cdot \frac{dx}{dt}$ and for ② $\frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$

Thus, the chain rule for $\frac{dw}{dt}$ consists of the <u>sum</u> of 2 <u>products</u>, i.e.

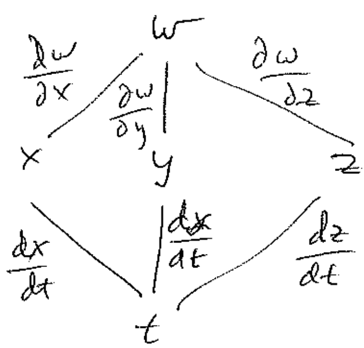
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}.$$

FUNCTIONS OF 3 VARIABLES

Thm 6 If $w = f(x, y, z)$ is diff'ble, and x, y, z are diff'ble functions of t, then f is a diff'ble function of t and we have the following version of the chain rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

DIAGRAM



EX 2 Given $w = xy + z$ $x = \cos t$, $y = \sin t$ $z = t$

$$\frac{\partial w}{\partial x} = y \quad \frac{\partial w}{\partial y} = x \quad \frac{\partial w}{\partial z} = 1 \quad \frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = 1$$

$$\therefore \frac{dw}{dt} = y(-\sin t) + x(\cos t) + 1 \cdot 1$$

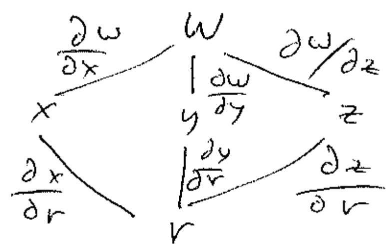
$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= 1 + \cos^2 t - \sin^2 t = 1 + \cos 2t$$

FUNCTIONS ON SURFACES p 824

Given $w = f(x, y, z)$, it could happen that each of x, y, and z could be a function of 2 (or more) variables, e.g. longitude & latitude. Thus, $x = x(r, s)$, $y = y(r, s)$, $z = z(r, s)$

We may need to know $\frac{\partial w}{\partial r}$, for example. A diagram gives:



But we also have an almost identical one for $\frac{\partial w}{\partial s}$!

Not that all the derivatives wrt r are partials!

∴ The chain rule is.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

cf Thm 7 p 824

EX Let $w = x^4 y + y^2 z^3$

cf EX 3 p 825

where $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s \sin t$

$\therefore$ $x = x(r, s, t)$!

Find $\frac{\partial w}{\partial s}$ when $r = 2, s = 1, t = 0$

$\Rightarrow$ $x = 2 \cdot 1 \cdot 1 = \underline{2}$ $y = 2 \cdot 1 \cdot 1 = \underline{2}$ $z = 4 \cdot 1 \cdot 0 = \underline{0}$

Using a diagram similar to before, we find

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

We have

$$\frac{\partial w}{\partial x} = 4x^3 y \qquad \frac{\partial w}{\partial y} = x^4 + 2yz^3 \qquad \frac{\partial w}{\partial z} = 3y^2 z^2$$

$$\frac{\partial x}{\partial s} = re^t \qquad \frac{\partial y}{\partial s} = 2rse^{-t} \qquad \frac{\partial z}{\partial s} = r^2 \sin t$$

$$\therefore \frac{\partial w}{\partial s} = 4x^3 y \cdot re^t + (x^4 + 2yz^3) \cdot 2rse^{-t} + 3y^2 z^2 \cdot r^2 \sin t$$

Evaluating at $r = 2, s = 1, t = 0$ and $x = 2, y = 2, z = 0$

we have

$$\frac{\partial w}{\partial s} = 4 \cdot 8 \cdot 2 \cdot 2 \cdot 1 + (16 + 0) \cdot 2 \cdot 2 \cdot 1 \cdot 1 + 3 \cdot 4 \cdot 0 \cdot 2^2 \cdot 0$$

$$= 128 + 64 + 0$$

$$= \underline{192}$$

CHAIN RULE AS DOT PRODUCT

p 828

When one has a function of several variables, e.g. $f(x, y, z, \ldots v)$, and each variable is a function of several variables, eg $x = x(p, q, r, \ldots t)$, one can see the chain rule as a dot product of 2 vector of partial derivatives.

The first vector is the collection of partials of f wrt each variable, eg.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \ldots \frac{\partial f}{\partial v}\right) \qquad [= \nabla f \text{ or gradient, 14.5, p 833}]$$

and The second vector is the collection of all the derivatives of the intermediate variable wrt the "final variable," eg

$$\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \frac{\partial z}{\partial p}, \ldots, \frac{\partial v}{\partial p}\right)$$

$$\therefore \frac{\partial f}{\partial p} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial p} + \ldots + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial p}$$

$$= \begin{pmatrix} \partial f/\partial x \\ \vdots \\ \partial f/\partial v \end{pmatrix} \cdot \begin{pmatrix} \partial x/\partial p \\ \vdots \\ \partial v/\partial p \end{pmatrix} = \nabla f \cdot \begin{pmatrix} \partial x/\partial p \\ \vdots \\ \partial v/\partial p \end{pmatrix}$$

IMPLICIT DIFFERENTATION AGAIN p 826

Thm 8 Given $F(x,y)=0$ (i.e. move all x,y expression to one side).

Let $w=F(x,y)=0$ and find $\frac{dw}{dx}$.

$\therefore \frac{dw}{dx}=0= F_x\frac{dx}{dx}+F_y\frac{dy}{dx} = F_x\cdot 1+F_y\frac{dy}{dx}$

$\Rightarrow F_y\frac{dy}{dx} = -F_x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$

p827 EX 5 Find $\frac{dy}{dx}$ if $y^2-x^2-\sin xy=0$.

Let $F(x,y)=y^2-x^2-\sin xy=0$

$\therefore \frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{-2x-y\cos xy}{2y-x\cos xy}$

$= \frac{2x+y\cos xy}{2y-x\cos xy}$