PARTIAL DERIVATIVES (14.3) p 810

In 2 dimensions, derivatives are useful concepts to give information about rate of change, easily seen as the slope of a curve.

We would like to extend this concept to 3-dimensions and be able to speak about slopes of surfaces.

In 2 dimensions, we could only approach a point along the x-axis (albeit from the right or the left, but still along only 1 axis).

In 3 dimensions, there are many paths toward a point on a surface. Thus, there will be several derivatives. In general we can speak of a directional derivative in a certain direction. More on this later in 14.5

Now we will focus on 2 basic directions: parallel to the x-axis and parallel to the y-axis.

If we let x change (move it ∥ to the x-axis) and fix y, we get the partial derivative of a function f with respect to x. If $z = f(x,y)$, we denote this as

p 811

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = D_x f$$

and define it as

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

<u>NOTE</u> The partial derivative symbol is called a "curly d". It corresponds to the italic small Russian d.

p 811 Similarly if we allow y to change (move it ∥ to y-axis) and keep x fixed, we get the partial derivative of f wrt y. We denote this as

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = D_y f.$$

and define it as

$$f_y(x,y) = \lim_{\substack{h \to 0 \\ \Delta x}} \frac{f(x, y+h) - f(x,y)}{h \, \Delta x}$$

To find either partial derivative, we hold the <u>other</u> variable fixed, i.e. consider it a constant, and differentiate with respect to the "moving" variable.

Eg.

pg 819 #7

$$f(x,y) = \sqrt{x^2+y^2} \qquad f_x(3,4) \Rightarrow \text{keep } y \text{ fixed.}$$

$$= (x^2+y^2)^{1/2}$$

$$f_x(x,y) = \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}\Bigg|_{(3,4)} = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$$

Let's also find $f_y(x,y)$ at $(3,4)$

$$f_y(x,y) = \tfrac{1}{2}(x^2+y^2)^{-1/2}\, 2y = \frac{y}{\sqrt{x^2+y^2}}\Bigg|_{3,4} = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$$

NOTE That $f_x(x,y) \neq f_y(x,y)$

cf EX4
p 813

Sometimes we need to use implicit differentiation.

$xy + yz = xz$ Find $\frac{\partial z}{\partial x}$

$$\Rightarrow y + y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial x} + z$$

$$\Rightarrow (y-x)\frac{\partial z}{\partial x} = z - y$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{z-y}{y-x} = \frac{y-z}{x-y}$$

FUNCTIONS OF MORE THAN 2 VAR'S

p 814

In general, we extend the procedure for 2 variables to other independent variables, if they exist. In each case we hold all the other variables constant if we differentiate wrT one of them.

The partial derivative $\frac{\partial u}{\partial x_i}$ if $u = f(x_1, x_2, \ldots, x_n)$

is $$\frac{\partial u}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1 \ldots, x_{i-1}, x_i + h, x_{i+1} \ldots x_n) - f(x_1 \ldots x_i \ldots x_n)}{h}$$

HIGHER DERIVATIVES p815

As with functions in the plane, we can take derivatives of derivatives. Unlike functions in the plane, we need to worry about which variables we are differentiating w.r.t. As a result there are 4 possible 2nd partial derivatives

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{yx} = \frac{\partial^2 f}{\partial x\, \partial y}$$

NOTE: Interpret $\frac{\partial^2 f}{\partial x \partial y}$ as $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ Thus, $\frac{\partial f}{\partial y}$ is done first

and f_{yx} as $(f_y)_x$. Thus, f_y is also done first.

<u>EG</u> Find all 2nd derivatives of

$$f(x,y) = x^4 - 3x^2y^3$$

cf EX 9 p 816

$$\frac{\partial f}{\partial x} = 4x^3 - 6xy^3 \qquad \frac{\partial f}{\partial y} = -9x^2y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 6y^3 \qquad \frac{\partial^2 f}{\partial y^2} = -18x^2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -18xy^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -18xy^2$$

NOTE: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

This will always occur if $f(x,y)$ is continuous near the point at which the derivative is evaluated.

See <u>Clairaut's Thm</u> (Thm 2 p 816)

PARTIAL DIFFERENTIAL EQUATIONS

Differential Equations that contain partial derivatives are called "Partial differential equations" or PDEs and are frequently used to describe physical laws. Eg., Laplace's equation

cf p820 before prob # 73

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is used to model heat conduction, fluid flow, etc.

cf p821 before prob # 81

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of waves of various sorts, eg. ocean, sound, light, violin string.

It is easy verify that certain functions satisfy PDEs. It is more difficult to determine a function given the PDE. cf MA 144

Eg. $u = x^2 - y^2$ and Laplace's eq.

cf # 73

$u_x = 2x \Rightarrow u_{xx} = 2$

$u_y = -2y \rightarrow u_{yy} = -2$

$$u_{xx} + u_{yy} = 2 + (-2) = 0$$ // Laplace is satisfied.