FUNCTIONS OF SEVERAL VARIABLES (14.1) p793

Quadric surfaces are one example of functions of several variables, if the value of z is uniquely determined by x and y.

In general, given an ordered collection of n real numbers $x_1, x_2, \dots, x_n$, a real valued function f is a rule assigning a value w to each set of values of $x_1 \dots x_n$

i.e $$w = f(x_1, x_2, \dots x_n).$$

As in the case of a function of 1 variable, the output variable w is called the dependent variable and the input variables $x_1, \dots x_n$ are called independent variables

We can determine the domain and range of a function of several variables in the same way as was done for a function of a single variable p794

(cf (1.1) p1)

LEVEL CURVES (pp 795-6)

Given a function of 2 variables, e.g. $z = f(x,y)$, if we fix the value of z to some constant K, we have a curve on the surface determined by f whose z-value ("height") is a constant.

Curves of this sort are called <u>level curves</u> or <u>contour curves.</u> (p 796)

Typical examples include <u>topographic maps</u> and <u>isothermal maps.</u>

If one "walks" along a level curve, one's "height" remains constant.

NOTE:

If curves each represent a value a constant difference from the next curve (eg, curves represent heights of 5, 10, 15, 20, ...) then the closer these curves are to one another, the steeper the surface is at that part. The farther apart they are, the flatter the surface.

EXAMPLES

= Contour map
= Iso-Thermal map
= Plot 3D

NOTE:

Given a function of 3 variables, eg. $w = f(x, y, z)$, the equivalent, in 3 dimensions, of a level curve is a level surface.

See p 796

I.e., set w equal to a constant k and the resulting 3D figure has a constant ("level") value on its "surface", $f(x, y, z) = K$.

WHERE ARE WE GOING?

Now that we have defined functions in higher dimensions, we want to do the same things for these functions as we did for $y = f(x)$ in Calc I, eg find slope, max/min points.

First we need to extend concepts to higher dimensions, eg. limits, continuity + derivatives.

16.2 Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \qquad \text{and} \qquad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

TABLE 1 Values of $f(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

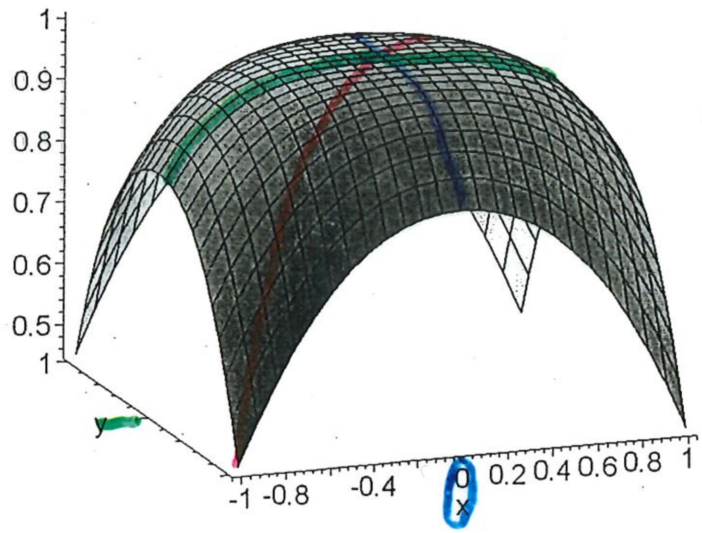
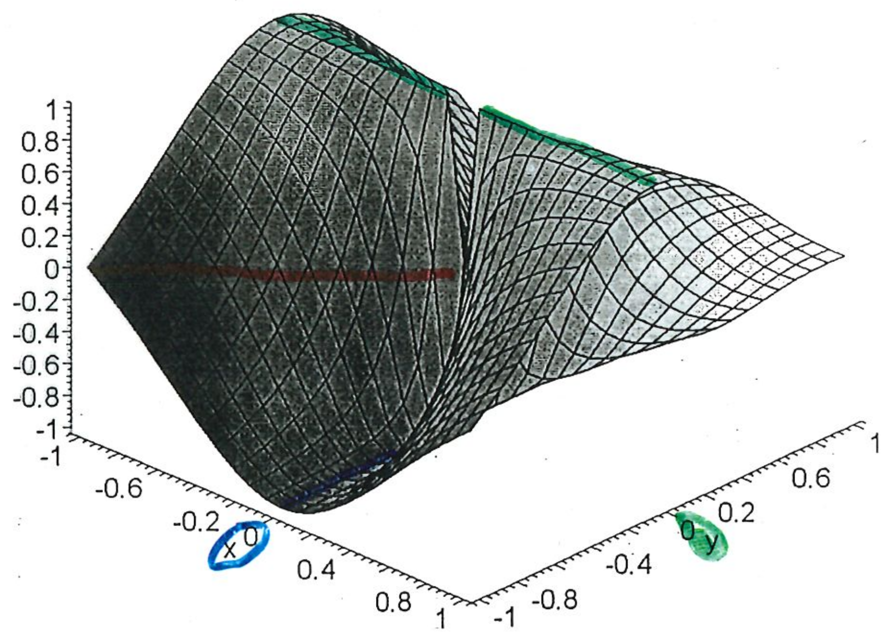


TABLE 2 Values of $g(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000



DEFINITION OF LIMIT

We need a definition of a limit in 3D that leads to a unique value, if it exists at all, no matter which path is taken to the desired location.

In the 2D case, we can either come to a point (x-value) from the right (value $> x$) or from the left (value $< x$). In the 3D case, we can come to a point identified by (x, y) from any direction of the complete $360°$ circle along a straight line or on any not-so-straight line. Thus we get the following:

pg 802

Def Let f be a function of 2 variables. Let the Domain of f include points close to (a, b).
We say the limit of $f(x,y)$ as (x,y) approaches (a,b) is L if
for every $\epsilon > 0$, there is a $\delta > 0$ s.t.
when (x,y) is δ-close to (a,b), i.e. $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$
Then $f(x,y)$ is ϵ-close to L, i.e. $|f(x,y) - L| < \epsilon$.

We write
$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$
or
$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = L$$
or
$$f(x,y) \to L \text{ as } (x,y) \to (a,b)$$

NOTE: "closeness" is determined only in terms of distance and not in terms of the path.

cf p806 "2-path Test"

Thus if $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ on path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ on path C_2 and $L_1 \neq L_2$, then the limit of $f(x,y)$ at (a,b) does not exist!

If you have 2 functions, the limit of the sum, product, difference, quotient is the sum, etc of the limits as in the 2 dim case. See Sect (2.2) Thm 1, p 69 and Sect (14.2) Thm 1, p 803

EXAMPLES

Eg 1 $\lim_{(x,y)\to(5,-2)} (x^5+4x^3y-5xy^2)$

Note that if we hold either x or y constant, the remaining function is continuous & without any "problem points". In fact, it is a polynomial in 2 variables and like its 2 D counterpart, it is "continuous" & The limit always exists.

Thus $\lim (x^5+4x^3y-5xy^2) = 5^5+4\cdot5^3\cdot(-2)-5(5)(-2)^2$
$= 3125-1000-100 = \underline{2025}$

Eg. 2 $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Note that the limit point is when x & y are both 0.
But in this case The denom is 0. Thus (0,0) is a problem point.
This suggest we should be careful at the origin.
Approach The origin in 2 different ways: along x-axis & along y-axis.
Along x-axis, we have $y=0$. Thus. limit reduces to

$$\lim_{x\to 0} \frac{x^2}{x^2} = \underline{1}$$

Along y-axis, we have $x=0$. Thus limit reduces to

cf "2-path Test" p 806

$$\lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = \underline{0}$$

Limits are different along 2 diff paths ∴ no limit exists.

CONTINUITY (p 805)

(2.5) p 93 The definition for continuity in 3D is similar to that of continuity for a function in 2D!! We need 3 conditions verified.

<u>Def</u> a function f of 2 variables is called <u>continuous at (a,b)</u> if

1) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists (and is finite)

2) $f(a,b)$ exists (and is finite)

3) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

<u>Def</u> In gen, we say f is <u>continuous on D</u> if f is continuous at every point (a,b) in D.

<u>Note</u> As in the 2D case, a polynomial of 2 variables is continuous.

<u>Eg</u> ① $f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 1$ is continuous. (polynomial)

Ex 5 p 805 ② $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Choose a "direction" line $y = mx$ (corresponding to some z-value) on the surface.

Then $$f(x,y)\Big|_{y=mx} = \frac{2x(mx)}{x^2+(mx)^2} = \frac{2mx^2}{x^2+m^2x^2}$$

$$= \frac{2mx^2}{(1+m^2)x^2} = \frac{2m}{1+m^2}$$

Thus $f(x,y)$ has the limit value of $\frac{2m}{1+m^2}$ along the line $y = mx$ for every $\underline{m}$!

I.e. if $m=1$, $f(x,y) = \frac{2\cdot 1}{1+1} = \frac{2}{2} = 1$

and if $m=3$, $f(x,y) = \frac{2\cdot 3}{1+9} = \frac{6}{10} = \frac{3}{5}$

$\therefore$ NO single limit value as $(x,y) \to (0,0)$!

FUNCTIONS OF SEVERAL VARIABLES

p 807

Concepts related to 2 D or 3 D functions can be extended to functions of 3 or more variables.

In particular The concepts of limit and continuity can be extended.