EQUATIONS OF LINES AND PLANES

I – LINES

Remember that 2 vectors are parallel if $\vec{v_1} = K\vec{v_2}$ for some scalar K.

Let L be a line in 3-space passing through a given point $P_0(x_0, y_0, z_0)$. Suppose L is parallel to a non-zero vector $\vec{v} = A\vec{i} + B\vec{j} + C\vec{k}$. Therefore, for any point $P(x, y, z)$ on the line L, the vector $\overrightarrow{P_0P}$ is parallel to the vector $\vec{v}$. $\therefore$ $\overrightarrow{P_0P} = t\vec{v}$ for some constant t. We can equate the corresponding components, getting:

$$\left.\begin{aligned} x - x_0 &= tA \\ y - y_0 &= tB \\ z - z_0 &= tC \end{aligned}\right\} \text{parametric equations of a line.}$$

or $x = x_0 + tA$, $y = y_0 + tB$, $z = z_0 + tC$

Solve each of these equations for t to obtain:

$$\frac{x - x_0}{A} = t \ ; \quad \frac{y - y_0}{B} = t \ ; \quad \frac{z - z_0}{C} = t$$

This gives us the Cartesian or standard equation of a line in 3-space.

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

where (x_0, y_0, z_0) is a point on the line and $A, B, C \neq 0$.

EX

Find a line through $(2,-9,5)$, $\parallel$ to
$\vec{v} = 3\vec{i} - \vec{j} + 4\vec{k}$. //

We get $A=3$, $B=-1$, $C=4$

Thus the line is:

$$\frac{x-2}{3} = \frac{y+9}{-1} = \frac{z-5}{4} \quad //$$

or in parametric form $\frac{x-2}{3} = t \Rightarrow x-2 = 3t \Rightarrow x = 3t+2$
and $y = -t-9$
$z = 4t+5$

II - PLANES

Let $P_0(x_0, y_0, z_0)$ be a point on a plane.

Let $\vec{N} = A\vec{i} + B\vec{j} + C\vec{k}$ be a normal vector (i.e, perpendicular to a plane). Therefore, for any point $P(x,y,z)$ on the plane, the vector $\overrightarrow{P_0P}$ must be perpendicular to $\vec{N}$. Therefore, $\vec{N} \cdot \overrightarrow{P_0P} = 0$

Now, $\overrightarrow{P_0P} = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$

So $\vec{N} \cdot \overrightarrow{P_0P} = A(x-x_0) + B(y-y_0) + C(z-z_0)$
and this $= 0$ for every point $P(x,y,z)$ on the plane.

Thus
$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$
is the equation of a plane $\perp \vec{N}$.

Alternate form is
$$Ax + By + Cz = D \quad [= Ax_0 + By_0 + Cz_0]$$

VECTOR NORMAL TO A PLANE

Let's go the other direction. Let

$$Ax + By + Cz = D$$

be the equation of a plane. If we find any point $P_0(x_0, y_0, z_0)$ on the plane, then the following must be true, that

$$Ax_0 + By_0 + Cz_0 = D$$

Subtracting these 2 equations, we get

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

which is

$$(A\vec{i} + B\vec{j} + C\vec{k}) \cdot (\overrightarrow{P_0P}) = 0$$

Thus, since the vector $\overrightarrow{P_0P}$ is in the plane, the other vector $A\vec{i} + B\vec{j} + C\vec{k}$ must be <u>normal</u> to $\overrightarrow{P_0P}$ for any point P_0 in the plane. Therefore, it is normal to the plane.

E.G. If $3x + 4y - z = 10$ is the equation of a plane, then

$3\vec{i} + 4\vec{j} - \vec{k}$ is <u>a</u> vector normal to the plane.

APPLICATIONS / PROBLEMS

cf EX 11 p737 Find the distance from a point to a plane.

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Point $= (2, -3, 4)$

Plane : $x + 2y + 2z = 13$

This is an alternate approach to method of book, which looks at the length of a vector projection

STEP 1 From the equation of the plane, we know that $\vec{N} = \vec{i} + 2\vec{j} + 2\vec{k}$ is a vector normal to the plane.

Given a vector and a point, we know that

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2}$$

is $\parallel$ to the vector $\vec{N}$ and goes through point $(2, -3, 4)$.

$\therefore$ This line also is normal to the plane. (END STEP 1)

STEP 2 The shortest distance from $(2, -3, 4)$ to the plane is along this line. Therefore, we must find the point where the line and plane intersect.

Let t be the common ratio of the quotients in the line equation, i.e. $\frac{x-2}{1} = t = \frac{y+3}{2} = \frac{z-4}{2}$

This leads to $x = t+2$, $y = 2t-3$, $z = 2t+4$

If we substitute these into the equation for the plane, we get

$$(t+2) + 2(2t-3) + 2(2t+4) = 13$$

$$t+2+4t-6+4t+8 = 13$$

$$9t+4 = 13$$

$$\Rightarrow 9t = 9 \quad \Rightarrow t=1$$

$\therefore$ at point of intersection,

$$x = (1)+2 = 3$$

$$y = 2(1)-3 = -1$$

$$z = 2(1)+4 = 6$$

(END STEP 2)

STEP 3 Thus, we want to find the distance between $(2,-3,4)$ and $(3,-1,6)$

using the distance formula

$$d = \sqrt{(2-3)^2+(-3+1)^2+(4-6)^2}$$

$$= \sqrt{(-1)^2+(-2)^2+(-2)^2}$$

$$= \sqrt{1+4+4} = \sqrt{9} = \underline{\underline{3}}$$

p 738 EX. 12 Find the angle between 2 planes.

<u>METHOD</u>: The angle between 2 planes is the same as the angle between 2 normal vectors to the planes, $\vec{N}_1$, $\vec{N}_2$.

We know $\vec{N}_1 \cdot \vec{N}_2 = |\vec{N}_1||\vec{N}_2| \cos\theta$

$$\therefore \theta = \arccos \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|}$$