THE VECTOR PRODUCT OF 2 VECTORS (12,4) p. 726

The "vector" product or "cross" product of 2 vectors gives a vector result, in comparison to the dot product.

Def $\vec{A} \times \vec{B} = \vec{n} \cdot |\vec{A}|\,|\vec{B}| \sin\theta$

where θ is the angle between $\vec{A}$ and $\vec{B}$ $(0 \le \theta \le \pi)$ and $\vec{n}$ is a unit vector perpendicular to the plane in which $\vec{A}$ and $\vec{B}$ lie.

The direction of $\vec{n}$ is the direction that a right-handed screw advances as it is turned from $\vec{A}$ to $\vec{B}$. This is called the right hand rule.

$\Rightarrow$ Take your right hand and rest it on vector $\vec{A}$ with the fingers curved in the direction of θ (toward $\vec{B}$). Your thumb points in the direction of $\vec{n}$.

NOTE: If $\vec{A}$ and $\vec{B}$ are interchanged, $\vec{n}$ changes direction,

$$\therefore \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Thus the cross product is NOT commutative.

Also NOTE: If $\vec{A} = \vec{B}$, then $\theta = 0$ $\therefore \sin\theta = 0$

$$\therefore \vec{A} \times \vec{B} = \vec{0}.$$

CROSS PRODUCTS OF BASIC VECTORS

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Applying the "right hand rule" to the basic vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$, we get the following:

$$\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$$

$$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$$

$$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}.$$

PHYSICAL REPRESENTATION

fig 12.30
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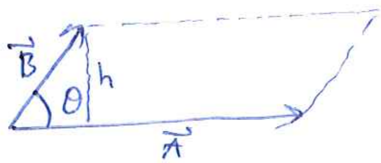

$h = |\vec{B}| \sin\theta$

The area of the parallelogram is $h \cdot |\vec{A}|$

$= |\vec{A}||\vec{B}| \sin\theta$

Now $\vec{n}$ in the cross product formula is a unit vector.

Thus $|\vec{n}| = 1$

Therefore the area $= |\vec{A}||\vec{B}| \sin\theta = |\vec{n}||\vec{A}|\vec{B}| \sin\theta$

$= |\vec{A} \times \vec{B}|$

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ALGEBRAIC LAWS

From the definition it is fairly trivial that scalar associativity holds, i.e.

$$(r\vec{A}) \times (s\vec{B}) = rs(\vec{A} \times \vec{B})$$

It can also be easily proven that distributivity also holds, i.e.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

and also

$$(\vec{B} + \vec{C}) \times \vec{A} = \vec{B} \times \vec{A} + \vec{C} \times \vec{A}$$

NOTE: you cannot reverse the order without introducing negatives!

CALCULATING CROSS PRODUCTS

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We would like to obtain a formula in terms of the parameters of the vectors $\vec{A}$ and $\vec{B}$ to obtain the cross product. We hope this would be similar (as to ease of calculation) to the formula obtained for the dot product.

In book. $\vec{A} = \vec{u}$, $\vec{B} = \vec{v}$

Using the algebraic laws (above), let us calculate $\vec{A} \times \vec{B}$ if $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
and $B = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{A} \times \vec{B} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

$$= a_1 b_1 \underbrace{\vec{i} \times \vec{i}}_{0} + a_1 b_2 \underbrace{\vec{i} \times \vec{j}}_{\vec{k}} + a_1 b_3 \underbrace{\vec{i} \times \vec{k}}_{-\vec{j}}$$

$$+ a_2 b_1 \underbrace{\vec{j} \times \vec{i}}_{-\vec{k}} + a_2 b_2 \underbrace{\vec{j} \times \vec{j}}_{\vec{0}} + a_2 b_3 \underbrace{\vec{j} \times \vec{k}}_{\vec{i}}$$

$$+ a_3 b_3 \underbrace{\vec{k} \times \vec{i}}_{\vec{j}} + a_3 b_2 \underbrace{\vec{k} \times \vec{j}}_{-\vec{i}} + a_3 b_3 \underbrace{\vec{k} \times \vec{k}}_{\vec{0}}$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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EXAMPLE

p. 728 Ex 3 Find the area of the triangle in 3-space with vertices P A(1, −1, 0), Q B(2, 1, −1), R C(−1, 1, 2).

① Represent 2 sides of the triangle by vectors.

$$\vec{a} = \overrightarrow{AB} = (2-1)\vec{i} + (1-(-1))\vec{j} + (-1-0)\vec{k}$$
PQ
$$= \vec{i} + 2\vec{j} - \vec{k}$$

$$\vec{b} = \overrightarrow{AC} = -2\vec{i} + 2\vec{j} + 2\vec{k}$$
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② From above, remember that the magnitude of the cross product is the area of the parallelogram bounded by 2 vectors.

$\therefore$ the area of a Triangle is $\frac{1}{2}|\vec{a} \times \vec{b}|$.

③ Use the computation formula above to first find $\vec{a} \times \vec{b}$.

cf EX 2

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= (4+2)\vec{i} + (2-2)\vec{j} + (2+4)\vec{k}$$

$$= 6\vec{i} + 6\vec{k}$$ (vector ⊥ to plane of triangle)

④ Now find magnitude and area.

$$|\vec{a} \times \vec{b}| = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$$

$\therefore$ area of triangle is $\frac{1}{2}$ this or $\underline{3\sqrt{2}}$.