SCALAR (DOT) VECTOR PRODUCT (12.3)

p 718

We have seen how to add/subtract vectors. Now let us examine the multiplication of vectors. There are 2 major ways to multiply vectors — one way gives a scalar as a result, and the other way gives a vector.

The scalar product uses a dot as the multiplication symbol and is also called the DOT product or INNER product.

The vector product uses a cross (x) as the multiplication symbol and is also called the CROSS product.

STANDARD

Def The scalar (dot) product of 2 vectors $\vec{A}$ and $\vec{B}$ is the number

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

where θ is the (smaller) angle between the 2 vectors (i.e., $0 \le \theta \le \pi$).

PROPERTIES

- By looking at the definition, it is obvious that

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

- Also, scalar multiplication is commutative, i.e.

$$c\vec{A} \cdot \vec{B} = \vec{A} \cdot c\vec{B}$$

- Because $\cos 0 = 1$, $A \cdot A = |A|^2$

or $|A| = \sqrt{A \cdot A}$

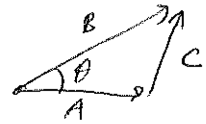
CALCULATION

Let $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{C} = \vec{B} - \vec{A}$ (i.e. $\vec{A} + \vec{C} = \vec{B}$)

In a picture



using the Law of Cosines (pg A-14)

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$$

$$\Rightarrow |\vec{A}||\vec{B}|\cos\theta = \frac{|\vec{A}|^2 + |\vec{B}|^2 - |\vec{C}|^2}{2}$$

$$\therefore \vec{A}\bullet\vec{B} = \frac{(a_1^2+a_2^2+a_3^2)+(b_1^2+b_2^2+b_3^2)-[(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2]}{2}$$

$$= \frac{a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2-b_1^2+2a_1b_1-a_1^2-b_2^2+2a_2b_2-a_2^2-b_3^2+2a_3b_3-a_3^2}{2}$$

p 719 $$= \underline{a_1b_1 + a_2b_2 + a_3b_3}$$

This is the standard formula used to compute the dot product.

It also enables us to calculate the angle between 2 vectors.

$$|\vec{A}||\vec{B}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

$$\Rightarrow \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{A}|\,|\vec{B}|}$$

Thm 1 p 718 $$\Rightarrow \theta = \arccos\left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{A}||\vec{B}|}\right]$$

By using the formula above, we can prove that the <u>distributive law</u> holds for dot products over addition, i.e.

$$\vec{A}\circ(\vec{B}+\vec{C}) = \vec{A}\circ\vec{B} + \vec{A}\circ\vec{C}$$

pg 721 BOX and

$$(\vec{A}+\vec{B})\circ\vec{C} = \vec{A}\circ\vec{C} + \vec{B}\circ\vec{C}$$

ORTHOGONALITY

<u>Def</u> If 2 vectors are perpendicular, i.e. if the angle between them is 90°, they are called <u>orthogonal</u>.

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<u>NOTE:</u> If $\vec{A}$ and $\vec{B}$ are orthogonal, $\vec{A}\circ\vec{B} = 0$.
(Converse is also considered true.)

PROJECTIONS

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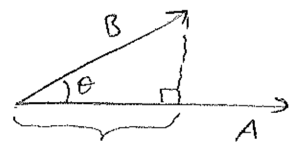
Take 2 vectors $\vec{A}$ and $\vec{B}$. If we draw $\vec{A}$ horizontally and let $\vec{A}$ and $\vec{B}$ have a common starting point, we can measure off a length in the direction of $\vec{A}$ by dropping a perpendicular from the tip of $\vec{B}$ onto $\vec{A}$.



A vector of this new length in the direction of $\vec{A}$ is called

<u>the vector projection of $\vec{B}$ onto $\vec{A}$</u> $= \text{proj}_{\vec{A}}\vec{B}$.

The <u>length</u> is <u>$|B| \cos\theta$</u> and is called the <u>scalar</u> projection of $\vec{B}$ onto $\vec{A}$ or the component of $\vec{B}$ in the direction of $\vec{A}$.

Note the following

$\Leftrightarrow \cos\theta = \frac{x}{|B|}$

$\Leftrightarrow x = |B| \cos\theta$

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta$$
$$= |\vec{A}| \left(|\vec{B}| \cos\theta \right)$$
$$= (\text{length of } \vec{A}) \, (\vec{B}\text{-component in } \vec{A}\text{-direction})$$
$$= (\text{length of } \vec{B}) \, (\vec{A}\text{-component in } \vec{B}\text{-direction})$$

WORK

p 723 In 6.6 W (work) = F (force) $\cdot$ d (distance).

In terms of vectors, $W = \vec{F} \cdot \vec{D}$ where $\vec{D} = \overrightarrow{PQ}$

EX 7 If $|\vec{F}| = 40$ newtons, $|\vec{D}| = 3\,m$ and $\theta = 60°$.

The work done by $\vec{F}$ in acting from P to Q (i.e. in the direction of $\vec{D}$) is:

$$W = \vec{F} \cdot \vec{D} = |\vec{F}| \cdot |\vec{D}| \cos\theta$$
$$= 40 \cdot 3 \cdot \cos 60°$$
$$= 120 \cdot \tfrac{1}{2} = \underline{60}$$

DIRECTION COSINES

cf. 12.3 Ex #15

Let $\vec{u}$ be a unit vector with initial point at the origin. Let α, β, γ be the angles $\vec{u}$ makes with the x, y and z axes.

Then
$$\vec{u} \cdot \vec{i} = |\vec{u}| \cdot |\vec{i}| \cos \alpha = \cos \alpha$$
$$\vec{u} \cdot \vec{j} = \cos \beta$$
$$\vec{u} \cdot \vec{k} = \cos \gamma$$

If $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$, then
$$\vec{u} \cdot \vec{i} = u_1 \quad \text{(via quick formula)}$$

In other words
$$u_1 = \cos \alpha$$
$$u_2 = \cos \beta$$
$$u_3 = \cos \gamma$$

These are called the <u>direction cosines</u>.

For an arbitrary vector $\vec{A}$,
$$\frac{\vec{A}}{|\vec{A}|} = \vec{i} \cos \alpha + \vec{j} \cos \beta + \vec{k} \cos \gamma$$

EXAMPLES

cf. p 724 #6

$\vec{A} = 3\vec{i} + 2\vec{j}$ $\quad \vec{B} = 5\vec{j} + \vec{k}$

Find $A \cdot B$, $|A|$, $|B|$, $\cos\theta$

$\vec{A} \cdot \vec{B} = 3 \cdot 0 + 2 \cdot 5 + 0 \cdot 1 = \underline{10}$

$|A| = \sqrt{9 + 4 + 0} = \underline{\sqrt{13}}$

$|B| = \sqrt{25 + 1} = \underline{\sqrt{26}}$

$$\cos\theta = \frac{10}{\sqrt{13}\sqrt{26}} = \frac{10}{13\sqrt{2}} = \frac{5\sqrt{2}}{13}$$

#16 Find the angle between the diagonal of a cube and one of its edges.

Diag of cube is $\parallel$ to $\vec{i} + \vec{j} + \vec{k}$ (if edges $\parallel$ to axes)

One edge is $\parallel$ to $\vec{i}$

$$\therefore \theta = \arccos \frac{1 + 0 + 0}{|\vec{i}| \cdot |\vec{i} + \vec{j} + \vec{k}|} = \arccos \frac{1}{1 \cdot \sqrt{3}}$$

$$= \arccos \frac{\sqrt{3}}{3}$$

cf #4 12 Given $\vec{A} = 2\vec{i} + 2\vec{j} + \vec{k}$, $\vec{B} = 2\vec{i} + 10\vec{j} - 11\vec{k}$

Find scalar projection of A onto B, i.e. the component of A in the direction of $\vec{B}$. ($= |\vec{A}| \cos\theta$)

$$\cos\theta = \frac{4 + 20 - 11}{\sqrt{4+4+1}\sqrt{4+100+121}} = \frac{13}{3 \cdot 15} = \frac{13}{45}$$

$$|\vec{A}| \cos\theta = 3 \cdot \frac{13}{45} = \frac{13}{15} \,//$$