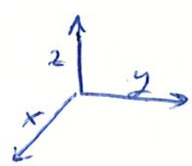
3-DIMENSIONS — SOLID GEOMETRY ((12.1)) 13

p704

For 3-dimensions, we frequently begin by dealing with a generalization of the Cartesian plane. Instead of 2 coordinate axes, we need 3 — x, y and now z. This approach is usually called the Cartesian coordinate system.

Usually this system is constructed of 3 axes, mutually at right angles, such that, if a right threaded screw is placed along the (positive) z-axis, it will advance if the head is twisted from the x-axis counter-clockwise toward the y-axis. This is called a right-handed system.



A point is designated by an ordered triple (x, y, z) in this Cartesian coordinate system.

An equation in a single variable denotes a plane (not a line), eg. $x = 3$. the y-z plane is $x = 0$.

The coordinate solid is divided into 8 octants. The one octant where all 3 coordinates are positive is called the 1st.

VECTORS (12.2) p 709

We have been dealing only in terms of magnitudes. However, often another concept is important in addition to magnitude — the concept of direction.

Def. A mathematical concept which expresses both direction and magnitude is called a vector.

Def A concept which expresses only a magnitude is called a scalar.

→ The most satisfactory algebra of vectors is based on representing vectors on the cartesian plane.

Def A vector is said to be free if it is "free" to move under parallel displacements.

→ Normally, we deal with "free" vectors. Thus, two vectors are equal if they have the same direction and the same magnitude. NOTE: free vectors are therefore NOT like vectors often used in physics and engineering which are usually tied to a point.

VECTORS IN 3-SPACE

713 In 3-space, you need 3 primary vectors when speaking about direction. The standard notation is:

$\vec{i}$ is the unit vector in the positive x-direction
$\vec{j}$ " " " " " " " y-direction
$\vec{k}$ " " " " " " " z-direction.

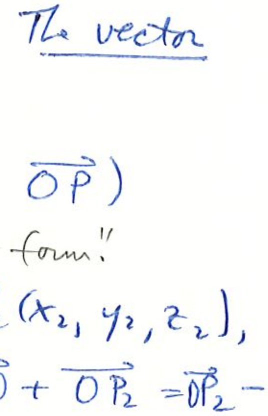
Pick a point $P(x, y, z)$ in 3-space. The vector from the origin to P is

$$\vec{R} = \vec{i}x + \vec{j}y + \vec{k}z. \quad (= \overrightarrow{OP})$$
$$= \langle x, y, z \rangle \text{ in "component form."}$$

Given 2 points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the vector from P_1 to P_2, $\overrightarrow{P_1P_2} = \overrightarrow{P_1O} + \overrightarrow{OP_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$
$= \vec{i}(x_2 - x_1) + \vec{j}(y_2 - y_1) + \vec{k}(z_2 - z_1)$.

To calculate the length of a vector in 3-space, we use the Pythagorean Thm twice. Look at $\vec{A} = a\vec{i} + b\vec{j} + c\vec{k}$

$$|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$$



A unit vector has length = 1.

Similarly, if $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ and $\overrightarrow{P_1P_2}$ is the vector from P_1 to P_2, then the distance between the 2 points is

p706

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

E.G. SPHERE

p706

If a sphere is defined to be a solid, equi-distant from a given point, then using the distance formula, we can get the equation:

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$$

This is the equation of a sphere with center at (x_0, y_0, z_0) and radius a. NOTE: $P(x,y,z)$ is the other "point" in the distance formula which varies on the surface of the sphere.

E.G. DIRECTION

p713

Def. The direction of a nonzero vector $\vec{A}$ in 3-space is the unit vector obtained by dividing $\vec{A}$ by its length.

i.e. direction of $\vec{A} = \dfrac{\vec{A}}{|\vec{A}|}$.

ex. Let $\vec{A} = 2\vec{i} - 3\vec{j} + 7\vec{k}$. $|\vec{A}|$ therefore equals $\sqrt{4+9+49} = \sqrt{62}$

$\therefore$ The direction of $\vec{A} = \dfrac{2\vec{i} - 3\vec{j} + 7\vec{k}}{\sqrt{62}}$

If the directions of $\vec{A}$ and $\vec{B}$ differ in sign, we say they point in opposite directions

Two vectors are parallel if they are scalar multiples of each other. Thus vectors pointing in opposite directions are parallel.

An alternative way of defining parallel vectors is given in 12.4,

EXAMPLES

cf EX4 p 706 Given $x^2+y^2+z^2+4x-4z=0$, find the center and radius of this sphere.

$$\Rightarrow x^2+4x \underline{+4} +y^2+z^2-4z \underline{+4} = \underline{8}$$

$$(x+2)^2+y^2+(z-2)^2 = (2\sqrt{2})^2$$

$\therefore$ center is $(-2, 0, 2)$ and radius $= 2\sqrt{2}$ //

cf EX1 p 711 #3 Find the length of $2\vec{i}+\vec{j}-2\vec{k}$

$$|2\vec{i}+\vec{j}-2\vec{k}| = \sqrt{4+1+4} = \sqrt{9} = 3$$ //

⇒ BE CAREFUL TO REMEMBER THE IMPLICATIONS OF 3D

① $x=10$ is line in 2D, but a plane in 3D.

② $x=2, y=3$ (12.1, p 707 #1) is a point in 2D but a line in 3D.

③ $x^2+y^2=4, z=3$ (12.1, p 705 Ex2) is a circle in the plane $z=3$.

④ $x^2+y^2+z^2=25, y=-4$ (12.1, p 707 #10) is a circle in the plane $y=-4$

$\Rightarrow x^2+(-4)^2+z^2=25 \Rightarrow x^2+z^2=25-16=9$

EQUALITY OF VECTORS

If $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k} = \langle a, b, c \rangle$

and $\vec{v} = a'\vec{i} + b'\vec{j} + c'\vec{k} = \langle a', b', c' \rangle$

Then $\vec{u} = \vec{v}$ iff.

$$a = a'$$
$$b = b'$$
$$c = c'$$

10-5 B

12-10B

VECTOR ALGEBRA

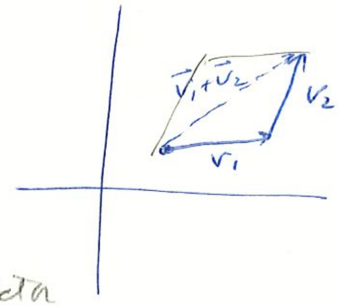
→ for simplicity, given info for 2D case

Addition

p 711

If $\vec{v}_1 = a_1\vec{i} + b_1\vec{j}$ and $\vec{v}_2 = a_2\vec{i} + b_2\vec{j}$

then $\vec{v}_1 + \vec{v}_2 = (a_1+a_2)\vec{i} + (b_1+b_2)\vec{j}$

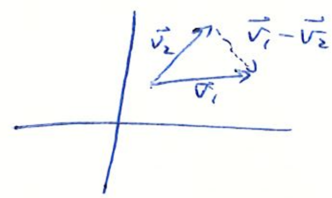


p 712 $\vec{v}_1 + \vec{v}_2$ is the diagonal of the parallelogram formed by $\vec{v}_1 + \vec{v}_2$ and is called the resultant vector

Subtraction

If $\vec{v}_1, \vec{v}_2$ are as above

then $\vec{v}_1 - \vec{v}_2 = (a_1 - a_2)\vec{i} + (b_1 - b_2)\vec{j}$



Magnitude / Length

p 710

(i.e. absolute value, distance)

$|\vec{v}| = |a\vec{i} + b\vec{j}| = \sqrt{a^2+b^2}$

Multiplication by Scalars

p 711

If c is any scalar,

then $c\vec{v} = (ca)\vec{i} + (cb)\vec{j}$

NOTE: $c\vec{v}$ is the same (opposite) direction as $\vec{v}$ but the magnitude is different.

Zero Vector

p 710

The zero vector, written $\vec{0}$, is any vector whose length is zero. $\vec{v} = a\vec{i} + b\vec{j} = \vec{0}$ iff $a = b = 0$.

MIDPOINT OF LINE SEGMENT

p714 Given $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

The midpoint M of the line segment joining P_1 and P_2

is $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right)$

E.G. EX 7 Given $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$,

$$M = \left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2} \right) = (5, 1, 2)$$

VECTORS IN NAVIGATION

p715

EX 8 A plane, flying due east at 500 mph, encounters a 70 mph tailwind blowing 60°N of east.
The plane holds its compass heading due east, but the wind changes the plane's speed and direction.
What are they?

Let $\vec{u}$ = vel. of plane alone and $\vec{v}$ = vel. of tailwind.

vel of plane wrt ground is $\vec{u}+\vec{v}$

$\vec{u} = \langle 500, 0 \rangle$, $\vec{v} = \langle 70\cos 60°, 70 \sin 60° \rangle = \langle 35, 35\sqrt{3} \rangle$

$\therefore \vec{u}+\vec{v} = \langle 535, 35\sqrt{3} \rangle$

Mag $|\vec{u}+\vec{v}|$ gives new ground speed $\therefore \sqrt{535^2 + (35\sqrt{3})^2} \doteq 538.4$

Direction θ is obtained by constructed a triangle with opp = j-comp and adj = i-comp $\therefore \theta = \arctan \frac{35\sqrt{3}}{535} \doteq 6.5°$