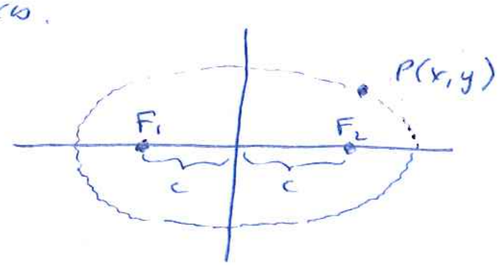
ELLIPSES (11.6 cont'd) p 685

<u>Def</u> An <u>ellipse</u> is the set of points in a plane whose distances from 2 fixed points have a constant sum.

<u>Def</u> The 2 given fixed points are called the <u>foci</u> (pronounced FOE-~~SIGH~~, Latin plural of FOCUS).

TO DERIVE AN EQUATION, start with the simplest form — locate the ellipse around the origin, with foci on the x-axis.



Let the foci be at $(-c, 0)$ and $(c, 0)$.

Let $P(x, y)$ be on the ellipse and the given constant be <u>$2a$</u>.

Then using the definition and the distance formula, we have

$$F_1P + F_2P = 2a$$

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

squaring $\Rightarrow$ $(x+c)^2+\cancel{y^2} = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+\cancel{y^2}$

$$\cancel{x^2}+2xc+\cancel{c^2} = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + \cancel{x^2} - 2xc + \cancel{c^2}$$

$$\cancel{4}xc - \cancel{4}a^2 = -\cancel{4}a\sqrt{(x-c)^2+y^2}$$

$$a - \frac{xc}{a} = \sqrt{(x-c)^2+y^2}$$

$$a^2 - 2\cancel{x}c + \frac{x^2c^2}{a^2} = x^2 - 2\cancel{x}c + c^2 + y^2$$

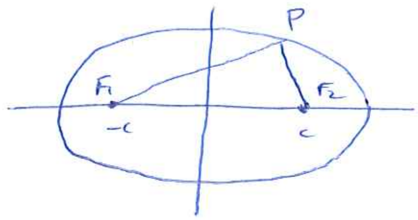
Skip algebra
(No time)

$$x^2\left(\frac{c^2}{a^2}-1\right) - y^2 = c^2 - a^2$$

$$\frac{x^2}{a^2}(c^2-a^2) - y^2 = c^2 - a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

RE-EXAMINE THE DIAGRAM



$F_1P + F_2P = 2a$

$F_1F_2 = 2c$

Obviously, the sum of 2 sides of a triangle is always greater than the third side.

$\therefore\ F_1P + F_2P > F_1F_2$

$\Rightarrow\ 2a > 2c \ \Rightarrow\ a > c \ \Rightarrow\ a^2 > c^2$

$\Rightarrow\ a^2 - c^2 > 0 \ \Rightarrow$ it has a <u>real</u> square root.

LET US CALL $+\sqrt{a^2-c^2} = b$

$\therefore$ STANDARD FORM OF ELLIPSE IS

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

PARAMETERS AND AXES

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

the ellipse intercepts the x-axis at $x = \pm a$

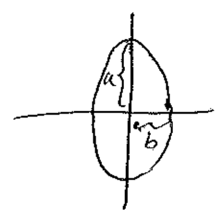
and intercepts the y-axis at $y = \pm b$

Assuming $a > b$, the longer axis is of length $\underline{2a}$ and is called the <u>major axis</u> pg 606

and the shorter axis is of length $2b$ and is called the <u>minor axis</u>

If the larger number is under y^2, then we have:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



ELLIPSES WITH CENTER NOT AT ORIGIN

We use the same technique as before (with the parabola).
Intuitively, the "center" of an ellipse is the point where the major and minor axes cross.

For simplicity, we took the center to be at $(0,0)$ for the examples above.

If the center is at (h,k), we make the standard adjustments (translation of axes), using

$$x' = x-h$$
$$y' = y-k$$

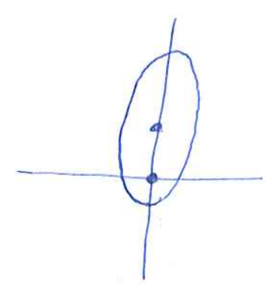
and obtain the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Algebraically, the equation of the ellipse is a quadratic in which BOTH x and y are QUADRATIC (no cross product) and the x^2 and y^2 terms have the SAME SIGN.

EX

$$C(0,2),\ F_1(0,0),\ a=3$$



Since F_1 and C are 2 units apart,
F_2 and C must also be 2 units apart
$\therefore\ F_2 = (0,4)$ (and $c=2$)

$$b = \sqrt{a^2-c^2} = \sqrt{9-4} = \sqrt{5}$$

$\therefore$ we have $$\frac{(x-0)^2}{5} + \frac{(y-2)^2}{9} = 1$$

or

$$9x^2 + 5(y-2)^2 = 45$$

EXAMPLES

EG 1 Reduce $5x^2 - 10x + 9y^2 - 54y + 41 = 0$ to standard form and indicate the major concepts.

cf #58 p691

$$5(x^2 - 2x + \underline{1}) + 9(y^2 - 6y + \underline{9}) = -41 + 5 + 81 = 45$$

$$5(x-1)^2 + 9(y-3)^2 = 45$$

$$\frac{(x-1)^2}{9} + \frac{(y-3)^2}{5} = 1 \quad \Rightarrow \quad \begin{array}{l} a = 3 \\ b = \sqrt{5} \end{array} \qquad \begin{array}{l} C = (1,3) \\ V = (4,3), (-2,3) \end{array}$$

$$\Rightarrow c^2 = a^2 - b^2 = 9 - 5 = 4 \Rightarrow c = 2$$

$$\Rightarrow F = (3,3), (-1,3)$$

$$[e = 2/3]$$

HYPERBOLAS (p 687)

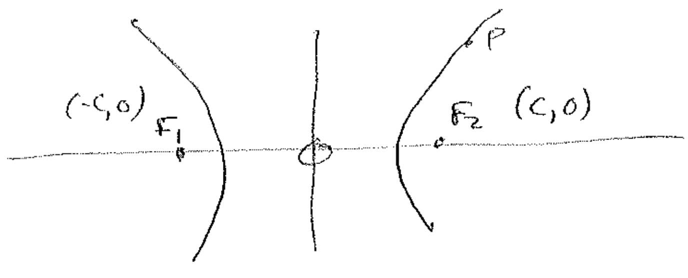
Def The hyperbola is the set of points in a plane whose distances from 2 fixed points have a constant difference.

Def The 2 fixed points are called foci.

Def (in terms of eccentricity) The hyperbola is the set of points such that the ratio of distances from a fixed point and a fixed line is a constant greater than 1.

If we take the foci as $(-c, 0)$, $(c, 0)$
and the constant (difference) as $2a$
and if we let $b^2 = c^2 - a^2$, (reverse of b^2 in ellipse)
The equation of a hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



ASYMPTOTES (cf sect 2.6 p104 ff)

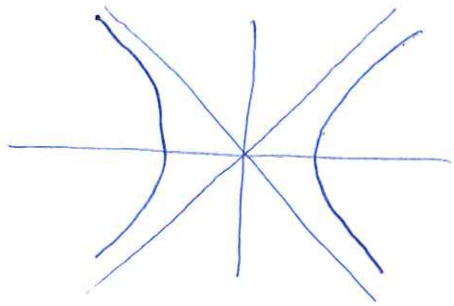
Look at the standard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Factor:

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 1$$

p 688 Both factors are straight lines — they form the asymptotes of the hyperbola.



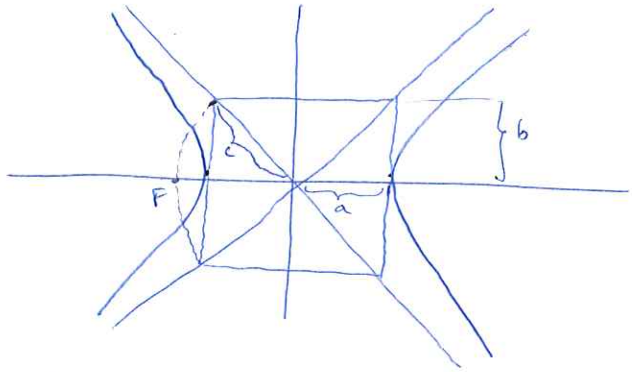
$\frac{x}{a} + \frac{y}{b} = 0$ is one line and $\frac{x}{a} - \frac{y}{b} = 0$ is the other

If we wish to designate them both simultaneously, we write

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

i.e. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

USE OF ASYMPTOTES



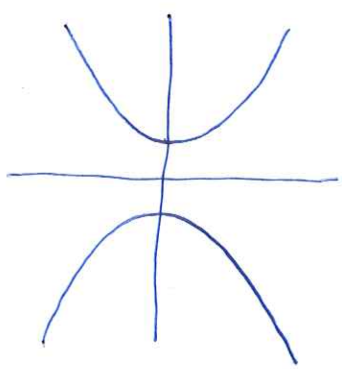
From the diagram, we can visualize a, b and c.

a is sometimes called the (length of the) <u>semi-transverse axis</u>
b is sometimes called the (length of the) <u>semi-conjugate axis</u>

<u>NOTE:</u> a is <u>NOT</u> necessarily bigger than b !

ROTATION BY 90°

As with the ellipse, if we rotate by 90°, we get a different standard equation.



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

EXAMPLES

$$4(y+3)^2 - 9(x-2)^2 = 1$$

$$\Rightarrow \frac{(y+3)^2}{(1/2)^2} - \frac{(x-2)^2}{(1/3)^2} = 1$$

$$\Rightarrow c^2 = a^2 + b^2 = \frac{1}{4} + \frac{1}{9} = \frac{9+4}{36} = \frac{13}{36}$$

$$\Rightarrow c = \frac{\sqrt{13}}{6}$$

Center: $(2, -3)$

Vertices: $(2, -3 \pm \frac{1}{2})$

Foci: $(2, -3 \pm \frac{\sqrt{13}}{6})$

Reduce to standard form

$$x^2 - y^2 - 2x - y + 1 = 0$$

$$x^2 - 2x + 1 - y^2 - y - \frac{1}{4} = -1/4$$

$$(x-1)^2 - (y+\tfrac{1}{2})^2 = -1/4$$

$$-\frac{(x-1)^2}{(1/2)^2} + \frac{(y+1/2)^2}{(1/2)^2} = 1$$

Center: $(1, -1/2)$

Vertices $(1, 0), (1, -1/2)$

Find the equation of a hyperbola with

$C(0,1)$, $\underbrace{V: (0,0), (0,2)}_{\Rightarrow a=1}$ and $\underbrace{F(0,-1), (0,3)}_{\Rightarrow c=2}$

$$b^2 = c^2 - a^2 = 4 - 1 = 3$$

$$\Rightarrow b = \sqrt{3}$$

$$\Rightarrow \frac{x^2}{1} - \frac{(y-1)^2}{3} = 1$$