HYPERBOLIC FUNCTIONS 7.3

Def.
$$\cosh u = \frac{e^u + e^{-u}}{2}$$ pronounced kosh

$$\sinh u = \frac{e^u - e^{-u}}{2}$$ pronounced sinch

Do we have any identity combining them as with normal trig functions?

$$\cosh^2 u - \sinh^2 u = \frac{(e^u+e^{-u})^2 - (e^u - e^{-u})^2}{4}$$

$$= \frac{\cancel{e^{2u}} + 2e^u e^{-u} + \cancel{e^{-2u}} - \cancel{e^{2u}} + 2e^u e^{-u} - \cancel{e^{-2u}}}{4}$$

$$= \frac{2+2}{4} = \frac{4}{4} = 1$$

$$\Rightarrow \underline{\cosh^2 u - \sinh^2 u = 1}$$

The other hyperbolic trig functions are defined as follows:

$$\tanh u = \frac{\sinh u}{\cosh u} \qquad \coth u = \frac{\cosh u}{\sinh u}$$

$$\operatorname{sech} u = \frac{1}{\cosh u} \qquad \operatorname{csch} u = \frac{1}{\sinh u}$$

The other identities can be derived as for normal trig functions.

$$\frac{\cosh^2 u}{\cosh^2 u} - \frac{\sinh^2 u}{\cosh^2 u} = \frac{1}{\cosh^2 u}$$

$$\Rightarrow \underline{1 - \tanh^2 u = \operatorname{sech}^2 u}$$

Similarly we get

$$\underline{\coth^2 u - 1 = \operatorname{csch}^2 u}$$

Graphs found on pg:

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OTHER IDENTITIES

Note:

$$\cosh x + \sinh x = \frac{e^{-x}+e^{x}}{2} + \frac{e^{x}-e^{-x}}{2} = \frac{\cancel{2}e^{x}}{\cancel{2}} = e^{x}$$

also

$$\cosh x - \sinh x = \frac{e^{x}}{2} + \frac{e^{-x}}{2} - \frac{e^{x}}{2} + \frac{e^{-x}}{2} = \frac{\cancel{2}e^{-x}}{\cancel{2}} = e^{-x}$$

$\therefore$ any combination of exponentials e^{x}, e^{-x} can be put in terms of combinations of <u>sinh x</u> and <u>cosh x</u>.

$$\sinh(x+y) = \frac{e^{x+y}-e^{-x-y}}{2} = \frac{e^{x}e^{y}-e^{-x}e^{-y}}{2}$$

$$\sinh x \cosh y + \cosh x \sinh y = \frac{e^{x}-e^{-x}}{2}\cdot\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}+e^{-x}}{2}\cdot\frac{e^{y}-e^{-y}}{2}$$

$$= \frac{e^{x}e^{y}+e^{x}e^{-y}-e^{-x}e^{y}-e^{-x}e^{-y}+e^{x}e^{y}-e^{x}e^{-y}+e^{-x}e^{y}-e^{-x}e^{-y}}{4}$$

$$= \frac{2e^{x}e^{y}-2e^{-x}e^{-y}}{4} = \frac{e^{x}e^{y}-e^{-x}e^{-y}}{2}$$

two are equal

$\therefore \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ (same as with circ.

other sum formulae can be derived.

eg. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Note: circular function have a minus here.

NEGATIVE ANGLES

$$\cosh(-x) = \frac{1}{2}\left(e^{-x} + e^{-(-x)}\right) = \cosh x$$

$$\sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{-(-x)}\right) = -\frac{1}{2}\left(e^{x} - e^{-x}\right) = -\sinh x$$

$\Rightarrow$ we get same identities as for circular trig functions.

DIFFERENCES

Hyperbolic functions are NOT periodic (since e^x is not).

Ranges of corresponding functions vary greatly.

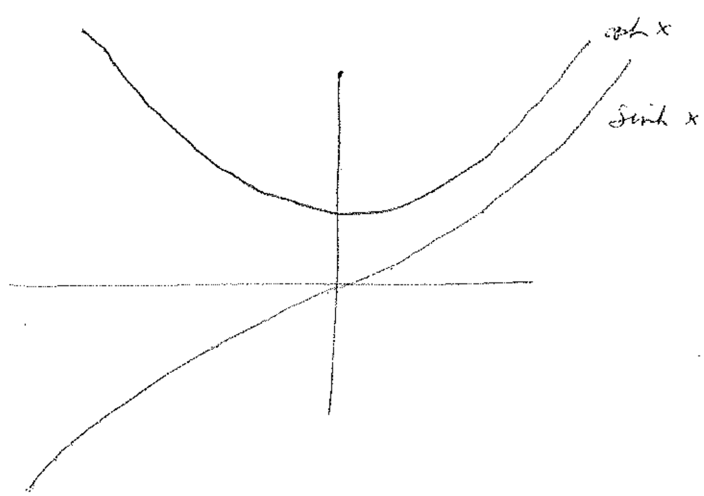
e.g. $\sin x$ goes from +1 to −1 oscillating

$\sinh x$ goes from $+\infty$ to $-\infty$ increasing

maybe part of 30-5

$\cos x$ oscillates between +1 and −1

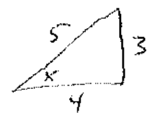
$\cosh x$ goes from $+\infty$ to 1 to $+\infty$



USING FORMULAS

For circular trig functions, we know that if

$\sin x = 3/5$



then

$\cos x = 4/5$

$\tan x = 3/4$

We can do something similar with hyperbolic trig functions. (except - cannot draw any figure)

Suppose $\sinh u = -3/4$

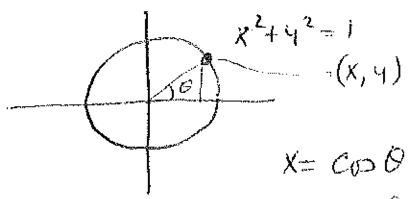
Then $\sinh^2 u = 9/16$

$\therefore$ by $\cosh^2 u - \sinh^2 u = 1$

we get $\cosh^2 u = 1 + \sinh^2 u = 1 + 9/16 = \frac{25}{16}$

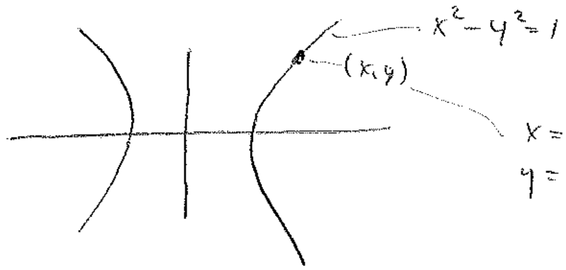
and $\cosh u = \frac{5}{4}$ since $\cosh u \geq 0$

COMPARING CIRCULAR + HYPERBOLIC TRIG FUNCTIONS



$x = \cos\theta$

$y = \sin\theta$



$x = \cosh u$

$y = \sinh u$

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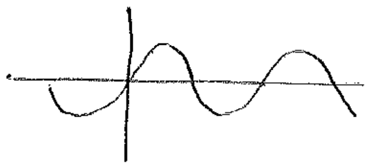
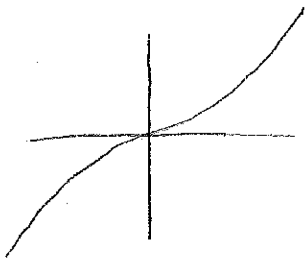
GRAPHS (p. ~~459~~ 570?)

$\sin x$



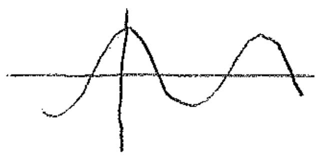
oscillates between +1, −1
periodic

$\sinh x$



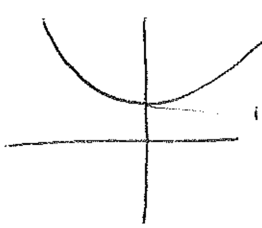
increasing from $-\infty$ to $+\infty$

$\cos x$



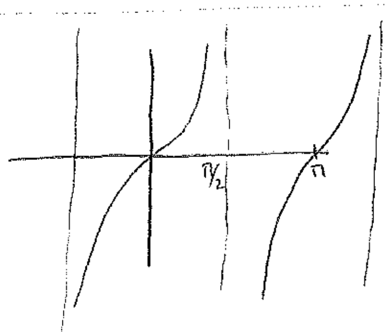
oscillates between 1 and −1

$\cosh x$



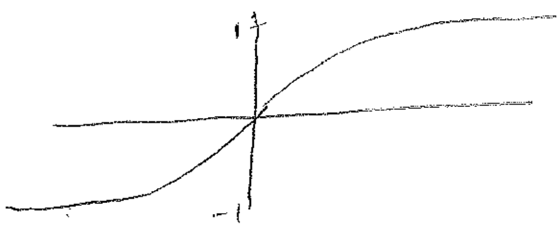
goes from $+\infty$ to 1 to $+\infty$

$\tan x$



$-\infty$ to ∞ periodic.

$\tanh x$



−1 to +1

DERIVATIVES OF HYPERBOLIC TRIG FUNCTIONS

Thm 1 $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$.

Proof

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left[\frac{e^u - e^{-u}}{2}\right] = \frac{e^u - (-e^{-u})}{2}\frac{du}{dx}$$

$$= \frac{e^u + e^{-u}}{2}\frac{du}{dx} = \cosh u \frac{du}{dx} \quad //$$

Thm 2 $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$ [NO minus sign!].

proof $$\frac{d}{dx}(\cosh u) = \frac{d}{dx}\left[\frac{e^u + e^{-u}}{2}\right] = \frac{e^u - e^{-u}}{2}\frac{du}{dx} = \sinh u \frac{du}{dx} \quad //$$

Thm 3

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

proof

$$\frac{d}{dx}(\tanh u) = \frac{d}{dx}\left(\frac{\sinh u}{\cosh u}\right) = \frac{\cosh u \cosh u - \sinh u \sinh u}{\cosh^2 u}\frac{du}{dx}$$

$$= \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u}\frac{du}{dx} = \frac{1}{\cosh^2 u}\frac{du}{dx} = \operatorname{sech}^2 u \frac{du}{dx}.$$

The other rules can be derived similarly.

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

NOTE.

These rules are similar to the rules for circular trig functions (p. 126-29)

The "same" functions appear in corresponding rules

But the minus sign is different.

In hyp. case, minus signs go with reciprocal functions.

In circ. case, minus signs go with "co" functions.

INTEGRAL RULES

We get corresponding integration formulae from the derivative rules.

eg.

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

etc.

EXAMPLES

$$y = \sin^{-1}(\tanh x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \tanh^2 x}} \cdot \operatorname{sech}^2 x$$

$$= \frac{1}{\sqrt{\operatorname{sech}^2 u}} \cdot \operatorname{sech}^2 x$$

$$= \frac{1}{\operatorname{sech} u} \operatorname{sech}^2 x = \operatorname{sech} x \,//$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} dx$$

$$= \int \frac{\sinh x}{\cosh x} dx \qquad u = \cosh x, \quad du = \sinh x \, dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\cosh x| + C \,//$$

$$\int \cosh^2 5x \, dx = \frac{1}{2} \int \cosh 10x + 1 \, dx$$

$$= \frac{1}{2} \frac{\sinh 10x}{10} + \frac{1}{2} x + C$$

$$= \frac{\sinh 10x}{20} + \frac{x}{2} + C \,//$$