Math 13-3 (10:30 AM)—Fall, 2024 | Final (2 hours)
D.C. Smolarski, S.J. | Friday, December 13, 2024, 9:10AM

perfect scores per problem

NOTES:

1. Problems can be written in the green book IN ANY ORDER, but please START each problem on a NEW PAGE (EITHER side) and label it properly.
2. PLEASE *label* (or underline or box in) all ANSWERS clearly.
3. Show your WORK — partial credit is possible only when all work needed to obtain an answer is presented legibly.
4. **NO CALCULATORS!**

19/32 1. (10) *(Taken from Midterm I.)* Find $\frac{dy}{dx}$ if $\arcsin y = \sinh x$. The answer should be all in terms of x.

20/32 2. (10) *(Taken from Midterm II.)* Find the Cartesian equation of the curve that has the following parametric equations: $x = \tan t$ and $y = \sin t$. *Do NOT include any trig or inverse trig function in the answer!*

18/32 3. (10) *(Taken from Midterm III.)* Given these points: A:(2,1,1), B:(1,2,1), C:(1,1,2).

   (a) Find the area of the triangle determined by the points.

   (b) Find the equation of a plane containing these same points.

23/32 4. (14) *(Taken from Spring 2024 Sample Final.)* Find the directional derivative $D_\theta z$ in the direction of 45° at the point (1,1,0) where $z = 2x^2 - 3xy + y$.

14/32 5. (15) Find the equation of the plane tangent to the surface $z = \ln(2x+y)$ at the point $(-1, 3)$.

22/32 6. (10) Given $w = x^2y^3 + y\cos x$ with $x = \ln(t^2)$ and $y = \sin 4t$, find $\frac{dw}{dt}$ using the chain rule. You may leave the final answer in terms of x, y, and t.

19/32 7. (14) Given that $f(x,y) = x^3y^4 - 2x\sqrt{y^3} + x^6 - \sin 3y$, find f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

⟹ *Turn over page for rest of exam!*

24/32 8. (14) A group of Broncos decide to go on a nature hike after finals in the Santa Cruz mountains. The hills they are climbing can be described by the equation $z = x^2 + xy + y^2$. They stop for lunch at a location where $x = -1$ and $y = 1$. One of the group, who lives on 8th floor Swig and is used to climbing stairs (especially when the elevators malfunction), decides to go mountain climbing and wants to go in the direction of steepest ascent.

EASIEST

(a) In terms of a vector, which direction should he go?

(b) Did you put your name on the cover of the green book?

(c) What is the rate of ascent (i.e., the slope of the mountain) in that direction?

(d) Take a long, deep breath—only two more problems!

11/32 9. (14) Find the locations of the local maximum, local minimum, and saddle points (if any) of the function $f(x, y) = x^3 + y^3 - 3xy + 15$.

10/32 10. (14) Use Lagrange multipliers to find the maximum and minimum locations and corresponding values of $f(x, y, z) = x + y + 2z$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 3$.

HARDEST

125 points total.

P.S. Have a restful Christmas break!

STATS

HI	125/125	MED 105	σ 19.23
LO	54/125	MEAN 102.4	

FORMULAS

Discriminant

$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$

Lagrange multiplier

$\nabla f = \lambda \nabla g$

Mt 13-3-10:30 AM - F24 - FINAL

1. $\arcsin y = \sinh x \Rightarrow \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \cosh x \Rightarrow \frac{dy}{dx} = \sqrt{1-y^2}\ \cosh x$

$\Rightarrow \frac{dy}{dx} = \cosh x \sqrt{1-\sin^2(\sinh x)} = \cosh x \sqrt{\cos^2(\sinh x)} = \underline{\cosh x \cdot \cos(\sinh x)}$

2. $x = \tan t \qquad y = \sin t \qquad$ since $\sin^2 t + \cos^2 t = 1$

$= \frac{\sin t}{\cos t} = \frac{y}{\cos t} \Rightarrow \cos t = \frac{y}{x} \qquad \Rightarrow \underline{y^2 + \frac{y^2}{x^2} = 1}$

3. A:(2,1,1) B:(1,2,1) C:(1,1,2)

a) $\vec{AB} = -\vec{i} + \vec{j}\ ,\ \vec{AC} = -\vec{i} + \vec{k}$ area of $\Delta = \frac{|\vec{AB} \times \vec{AC}|}{2} = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$

$= \frac{1}{2}\left|\vec{i}+\vec{j}+\vec{k}\right| = \underline{\frac{\sqrt{3}}{2}}$

b) $\vec{N} = \vec{AB} \times \vec{AC} = \vec{i} + \vec{j} + \vec{k}$

$\Rightarrow \underline{1(x-2) + 1(y-1) + 1(z-1) = 0} \Rightarrow x+y+z = 2+1+1 \Rightarrow \underline{x+y+z = 4}$

4. $z = 2x^2 - 3xy + y \qquad P_0(1,1,0)$

$z_x = 4x - 3y\,\big|_{P_0} = 4 - 3 = 1$

$z_y = -3x + 1\,\big|_{P_0} = -3 + 1 = -2$

$D_{45^\circ} z = z_x \cos 45^\circ + z_y \sin 45^\circ$

$= 1 \cdot \frac{\sqrt{2}}{2} + (-2)\frac{\sqrt{2}}{2} = \underline{\frac{-\sqrt{2}}{2}}$

5. $z = \ln(2x+y) \qquad P_0(-1,3)$

$z_x = \frac{2}{2x+y}\Big|_{P_0} = \frac{2}{-2+3} = 2$

$z_y = \frac{1}{2x+y}\Big|_{P_0} = \frac{1}{-2+3} = 1$

$z\big|_{P} = \ln(-2+3) = \ln 1 = 0$

Eq of plane $\perp$

$\underline{2(x+1) + 1(y-3) - 1(z-0) = 0}$

$\Rightarrow \underline{z} = 2x + 2 + y - 3 = \underline{2x + y - 1}$

6. $z = x^2y^3 + y\cos x \qquad x = \ln(t^2) \qquad y = \sin 4t$

$\frac{\partial z}{\partial x} = 2xy^3 - y\sin x \quad \frac{\partial z}{\partial y} = 3x^2y^2 + \cos x \quad \frac{dx}{dt} = \frac{1}{t^2}\cdot 2t = \frac{2}{t} \quad \frac{dy}{dt} = 4\cos 4t$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \underline{(2xy^3 - y\sin x)\left(\frac{2}{t}\right) + (3x^2y^2 + \cos x)(4\cos 4t)}$

7. $f(x,y) = x^3y^4 - 2x\sqrt{y^3} + x^6 - \sin 3y$

$f_x = 3x^2y^4 - 2y^{3/2} + 6x^5$ $\quad f_y = 4x^3y^3 - 3xy^{1/2} - 3\cos 3y$

$f_{xx} = \underline{6xy^4 + 30x^4}$ $\quad f_{yy} = \underline{12x^3y^2 - \frac{3}{2}xy^{-1/2} + 9\sin 3y}$

$f_{xy} = \underline{12x^2y^3 - 3y^{1/2}}$ $\quad f_{yx} = \underline{12x^2y^3 - 3y^{1/2}}$

8. $z = x^2 + xy + y^2 \qquad P_0(-1,1)$

$\frac{\partial z}{\partial x} = 2x + y \Big|_{P_0} = -2+1 = -1 \qquad \frac{\partial z}{\partial y} = x + 2y \Big|_{P_0} = -1+2 = 1$

a) $\vec{\nabla z} = (-1)\hat{i} + (1)\hat{j} = \underline{-\hat{i} + \hat{j}}$ $\Leftarrow$ direction

c) Rate $= |\vec{\nabla z}| = \sqrt{1+1} = \underline{\sqrt{2}}$

9, $f(x,y) = x^3 + y^3 - 3xy + 15$

$f_x = 3x^2 - 3y = 0 \Rightarrow y = x^2$ $\Rightarrow x^4 - x = 0 \rightarrow x(x^3-1) = 0 \Rightarrow x = 0$ or 1

$f_y = 3y^2 - 3x = 0 \Rightarrow y^2 - x = 0$

$x = 0 \Rightarrow y = 0$
$x = 1 \Rightarrow y = 1$

critical pts are $(0,0)$ & $\underline{(1,1)}$

$f_{xx} = 6x$
$f_{yy} = 6y$
$f_{xy} = -3$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 6x \cdot 6y - (-3)^2 = 36xy - 9$

Pt	D	f_{xx}	Conclusion	
(0,0)	−9	——	D < 0	$\Rightarrow$ saddle pt at (0,0)
(1,1)	27	$\Leftarrow f_{xx} > 0$		$\Rightarrow$ loc min at (1,1)

10. $f(x,y,z) = x + y + 2z \qquad g(x,y,z) = x^2 + y^2 + z^2 = 3 \longrightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 3$

$\nabla f = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \lambda \nabla g = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \Rightarrow \begin{matrix} 1 = 2\lambda x \\ 1 = 2\lambda y \\ 1 = 2\lambda z \end{matrix} \Rightarrow \begin{matrix} x = \frac{1}{2\lambda} \\ y = 1/2\lambda \\ z = 1/\lambda \end{matrix}$ $\quad = \frac{1+1+4}{4\lambda^2} = \frac{6}{4\lambda^2} = \frac{3}{2\lambda^2}$

$\Rightarrow 2\lambda^2 = 1 \Rightarrow \lambda^2 = \frac{1}{2} \Rightarrow \lambda = \pm \frac{\sqrt{2}}{2}$

$\therefore$ extreme values are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$

$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 2\sqrt{2} = 3\sqrt{2} \Leftarrow$ max at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right)$

$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 2\sqrt{2} = -3\sqrt{2} \Leftarrow$ min at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$