SEPARABLE DIFFERENTIAL EQUATIONS

In Section 7.2, we learned how to solve a *separable differential equation*; that is, a differential equation of the form

$$\frac{dy}{dt} = f(y)g(t).$$

The equation is called "separable" because we can separate the two variables, y and t in this case, to the two sides of the equation

$$\frac{dy}{f(y)} = g(t)dt$$

and then integrate both sides

$$\int \frac{dy}{f(y)} = \int g(t)dt.$$

Evaluating these two integrals, using the methods from Chapter 8 if necessary, yields y as an implicit function of t.

In this section we further explore this technique, which is called *separation of variables*, and use it to solve a small sampling of the many separable differential equations that naturally arise in science and economics.

EXAMPLE 1:

The differential equation

$$\frac{dy}{dt} = \frac{y^2+1}{t+1}$$

is separable ($f(y) = y^2+1$ and $g(t) = \frac{1}{t+1}$ here). So we solve for y by separating the variables:

$$\frac{dy}{y^2+1} = \frac{dt}{t+1}$$

and integrating

$$\int \frac{dy}{y^2+1} = \int \frac{dt}{t+1}.$$

Evaluating these integrals yields

$$\tan^{-1}(y) = \ln(t+1) + C \qquad \text{(remember the } C\text{!)}.$$

When possible, as it is here, we isolate y so that we have an explicit formula for y as a function of t :

$$y = \tan\left(\ln(t+1) + C\right).$$

(Note that the "$+C$" is now part of the input for the tangent function.)

EXAMPLE 2:

The differential equation

$$\frac{dy}{dt} = y + t$$

is *not* separable. Try as hard as you want: you will not be able to get all the y's on one side of the equation and all the t's on the other (unless you violate the rules of algebra, which is a *very* bad idea!). We can also see that the equation is not separable because $y + t$ cannot be written in the form $f(y)g(t)$, that is, a function of y times a function of t. The solution to this non-separable differential equation happens to be $y = Ce^t - (t+1)$, which one can determine using methods that are taught in Math 22 or AM 106. (However, you don't need those classes to plug this solution into the differential equation and verify that it satisfies the equation.)

INITIAL CONDITIONS AND UNIQUE SOLUTIONS

Whenever you solve a separable differential equation, you always get a "$+C$" after the integration, so, since C is arbitrary, you have an infinite number of solutions. To get a unique solution, an "initial condition" must be specified in addition to the differential equation. Although the initial condition is usually the value of the unknown, y, at $t = 0$, i.e.,

$$y(0) = y_0,$$

sometimes it is the value of y at another time, $t = t_0$, so we have

$$y(t_0) = y_0.$$

We still call this an "initial condition", even though the name makes less sense in this case. Sometimes we don't even have t (time) as the independent variable in our differential equation—we *still* use the term "initial condition".

There are two methods of using the initial condition to get the unique solution provided by separation of variables. You will want to master both because both are commonly used in science (as well as math) classes and textbooks. We demonstrate both methods in the example

$$\begin{aligned} y' &= te^y && \text{(separable differential equation)} \\ y(6) &= 3 && \text{(initial condition: } y = 3 \text{ when } t = 6\text{).} \end{aligned}$$

Note that it is implied here that y is a function of t, even though we only say $y' = te^y$, instead of $y'(t) = te^y$.

METHOD 1: INDEFINITE INTEGRATION

We begin by applying the standard method used in Section 4.1 to our differential equation:

$$\frac{dy}{dt} = te^y$$

$$\frac{dy}{e^y} = tdt$$

$$\int e^{-y} dy = \int tdt$$

$$-e^{-y} = \frac{1}{2}t^2 + C.$$

Next we substitute the initial condition ($y = 3$ at $t = 6$) into our last result to determine the value of C :

$$-e^{-3} = \frac{1}{2}6^2 + C \quad \Longrightarrow \quad C = -18 - \frac{1}{e^3}.$$

Finally, we substitute C back into our solution, and, since we can, we isolate y so that it is an explicit function of t :

$$-e^{-y} = \frac{1}{2}t^2 - 18 - \frac{1}{e^3}$$

$$y = -\ln\left(-\frac{1}{2}t^2 + 18 + \frac{1}{e^3}\right).$$

Note that this solution is only defined for times where $-\frac{1}{2}t^2 + 18 + \frac{1}{e^3} > 0$.

METHOD 2: DEFINITE INTEGRATION

We perform the same separation as in the first method, but then we integrate from the time of the initial condition, t_0, to an arbitrary final time, t_f. *Note that for the integral with respect to y this corresponds to integrating from $y(t_0)$ to $y(t_f)$.*

$$\frac{dy}{dt} = te^y$$

$$\int_{y(t_o)}^{y(t_f)} e^{-y} dy = \int_{t_0}^{t_f} tdt$$

$$\int_{3}^{y(t_f)} e^{-y} dy = \int_{6}^{t_f} tdt$$

$$-e^{-y} \Big|_{3}^{y(t_f)} = \frac{1}{2}t^2 \Big|_{6}^{t_f}$$

$$-e^{-y(t_f)} + e^{-3} = \frac{1}{2}t_f^2 - \frac{1}{2}6^2.$$

Since t_f is any arbitrary time, we can just replace it with t. Solving for $y(t)$ gives us

$$y(t) = -\ln\left(-\frac{1}{2}t^2 + 18 + \frac{1}{e^3}\right),$$

the same result as in the first method.

A GREAT QUESTION (THOUGH YOU MAY BE SORRY YOU ASKED IT!)

If you're a thoughtful, curious type of person, you may have wondered why, in the definite integration method, we didn't just integrate from t_0 to t instead of integrating from t_0 to t_f and then substituting t for t_f later. If we had done that, we would have had the expression

$$\int_6^t tdt,$$

which would not be correct notation (though many use it anyway) because as the t in the integrand varies, the t in the upper limit would also have to vary, which is not what we want to happen. If we want to integrate from 6 to t and avoid this problem, we must change the dummy variable in the integrand to something (actually, anything!) other than t; for example, we could have proceeded with the steps

$$e^{-y}dy = tdt$$

$$\int_3^y e^{-z}dz = \int_6^t sds.$$

Here z and s are our new dummy variables. Changing the dummy variables is not an uncommon practice—and it definitely yields the right answer—but it can be confusing when the original variables represent physical quantities, which is why many physical scientists avoid it.

APPLICATIONS (A.K.A. WORD PROBLEMS)

Since separable differential equations are so plentiful in the sciences, it is essential to get some practice solving these equations in the same contexts where you will encounter them in later courses. That means solving word problems, but if you use the following steps you should be fine, both now and in subsequent classes:

1. STAY CALM. (Ignoring this advice leads to half the problems people have with word problems.)

2. Determine which symbols in your problem represent constants and which symbols represent the two variables of interest.

3. Separate the two variables (constants can go on either side of the equation—it makes no difference in your final answer), integrate the separated equation, and use the initial condition to find a unique solution.

4. When possible, isolate the variable representing the unknown function; this unknown will often, though not always, depend on t (time).

EXAMPLE 3

Question: A gas is called "ideal" if it conforms to the relationship

$$PV = nRT$$

where P is the pressure, V is the volume of the gas, n is the number of molecules in the gas, R is a gas law constant, and T is the temperature. Using related rates, which you had the pleasure of meeting in Section 3.10, the above relationship can be differentiated with respect to time, t, and, for a fixed number of ideal gas molecules held at a constant temperature, we have the following relationship:

$$\frac{dV}{dt} = \frac{-nRT}{P^2}\frac{dP}{dt}. \tag{1}$$

Now assume the air in a balloon is ideal and maintains a constant temperature. If the pressure at $t = 0$ is 3 pressure units and the balloon shrinks at the rate

$$\frac{dV}{dt} = -t^3, \tag{2}$$

what is the pressure of the gas in the balloon as a function of time?

Answer: The question asks for $P(t)$, so the two variables for which we want to hunt are P and t. This leads us to look at equation (1). Since the number of molecules in a balloon is fixed and the temperature is constant, n and T, as well as R, are constants. $\frac{dV}{dt}$ is a variable but we can use (2) to express $\frac{dV}{dt}$ as a function of t. Substituting (2) into (1) gives us our differential equation

$$-t^3 = \frac{-nRT}{P^2}\frac{dP}{dt},$$

which is separable (a good sign!) The problem also gives us the initial condition

$$P(0) = 3.$$

Now we're ready to find the unique solution. We will now solve our resulting equation using the definite integral method (although the indefinite integral method will work just as well).

$$-t^3 dt = -nRT\frac{dP}{P^2}$$

$$\int_0^{t_f} t^3 dt = nRT\int_3^{P(t_f)} \frac{dP}{P^2}$$

$$\left.\frac{t^4}{4}\right|_0^{t_f} = nRT\left(-\left.\frac{1}{P}\right|_3^{P(t_f)}\right)$$

$$\frac{t_f^4}{4} = nRT\left(-\frac{1}{P(t_f)} + \frac{1}{3}\right).$$

Replacing t_f with t and isolating P gives our final answer:

$$P(t) = \frac{1}{\frac{1}{3} - \frac{t^4}{4nRT}}.$$

PROBLEMS

In **Problems 1–8** determine if the differential equation is separable. If it is separable, use separation of variables to determine an infinite number of solutions to the equation. (Recall that you will have an infinite number of solutions due to the presence of the arbitrary constant C.)

Express your answers in explicit form, so, for example, an implicit solution like $\ln(y+3) = x + C$ should be re-expressed into a form where y is isolated, like $y = e^{x+C} - 3$.

1) $\dfrac{dy}{dx} = \dfrac{x}{y^2}$

2) $y' = y^2 \sin x$

3) $\dfrac{dy}{dt} = y^2 + e^t$

4) $\dfrac{dy}{dt} = \dfrac{y^2}{y + \cos(t)}$

5) $(x+1)y' = \cos^2 y \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$

6) $\dfrac{dz}{dt} = \dfrac{z - 4t}{t - z}$

7) $\dfrac{df}{dt} = \dfrac{t(f^2 + f)}{e^t}$ $\quad f > 0$

8) $r' = r^2 t + r^2 + 9t + 9$

In **Problems 9–22** determine a unique solution of the separable differential equation that satisfies the given initial condition. (You can leave your solution in an implicit form. For example, the form $y^2 + y = 3x + 2$ is fine, as opposed to having to isolate y as an explicit function of x.)

9) $\dfrac{dy}{dt} = y^2$ $\quad y(1) = 2$

10) $\dfrac{dy}{dx} = \dfrac{2x}{1 + 2y}$ $\quad y(2) = 0$

11) $\sin(2x) + y' \cos(3y) = 0$ $\quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$

12) $r\,dr + \theta e^{-r} d\theta = 0$ $\quad r(0) = 1$

13) $\dfrac{dc}{dt} = c^3 \ln(t)$ $\quad c(e^2) = 1$

14) $\dfrac{dz}{dt} = \dfrac{1}{2z + t^2 z}$ $\quad z(0) = -2$

15) $\dfrac{dM}{dx} = M^2 - 5M + 6$ $\quad M(1) = 4$

16) $\dfrac{dg}{d\theta} = g \cos^3(\theta)$ $\quad g(0) = 3$

17) $\dfrac{df}{dt} = (f^2 - f) t e^{-t}$ $\quad f(0) = 13$

18) $\dfrac{d\rho}{ds} = s \cos(s) \csc^2(\rho)$ $\quad \rho\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$

19) $\dfrac{dh}{dp} = e^p \sin(p) \sec^3(h)$ $\quad h(0) = 0$

20) $\dfrac{dy}{dx} = \dfrac{(y + 2)^2}{\sqrt{4 - x^2}}$ $\quad y(1) = 2$

21) $\dfrac{d^2y}{dt^2} = \dfrac{dy}{dt}$ $\quad \dfrac{dy}{dt}(0) = 2, \quad y(0) = 3$ [Hint: Define $v(t) = \dfrac{dy}{dt}$ and solve the differential equation for $v(t)$ first. Then solve for $y(t)$.]

22) $\dfrac{d^2y}{dt^2} = \dfrac{3t^2}{2\frac{dy}{dt}}$ $\quad \dfrac{dy}{dt}(1) = -1, \quad y(1) = 2$ [See hint for problem 21.]

Problems 23–31 are adapted from undergraduate courses and texts in physics, chemistry, biology, and engineering. Each involves solving a separable differential equation. Remember to include units in your answers where appropriate.

23) **Conservation of Momentum** Early on in life you learn Newton's second law of motion: $F = ma$; that is, force equals mass times acceleration. Later on, you learn that this is a simplification of $\sum_{i=1}^{n} F_i = ma$; that is, the sum of all the forces acting on a body equals that body's mass times its acceleration. But this is also a simplification. The more general form is $\sum_{i=1}^{n} F_i = \frac{d(mv)}{dt}$. The term $\frac{d(mv)}{dt}$ is the derivative of momentum (which is defined as the mass times the velocity of a body) with respect to time, t. Only when the mass of the body stays constant do we have that $\frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$.

If there are no forces acting on a body, momentum is conserved, since $\frac{d(mv)}{dt} = 0$ implies the momentum stays constant over time. If both the mass and velocity are changing over time, then we can apply the product rule to obtain $m\frac{dv}{dt} = -v\frac{dm}{dt}$.

For example, the motion of a rocket ship in space is given by

$$m\frac{dv}{dt} = -u\frac{dm}{dt},$$

where m is the mass (in thousands of kg) of the rocket (which changes as fuel is consumed by the rocket), v is the velocity (in km/sec) of the rocket, and $-u$ is the velocity (in km/sec) relative to the rocket of the fuel exhaust ejected from the back of the rocket. If the initial mass, m, of the rocket is 5 and the initial velocity of the rocket is 0 (so $m = 5$ when $v = 0$), and $u = 5\sqrt{m}$, find the velocity of the rocket when the mass is 4. Remember to include the correct units of the velocity in your answer. Hint: you can "cancel" the dt's in the differential equation.

24) **Falling Bodies** If a body is thrown from a plane, there are two forces that

act upon it: the gravitational force, $-mg$, where m is the mass of the body and g is the acceleration due to gravity, and the drag force from air friction, which is well modeled by cv^2, where c is a constant and v is the velocity of the body.

From our discussion of motion in the first paragraph of the previous problem, we know that $ma = F_{\text{gravity}} + F_{\text{drag}}$, and so we have the following differential equation for the velocity $v(t)$:

$$m\frac{dv}{dt} = -mg + cv^2.$$

The value of g is 9.8 m/sec^2. Assume we have a body that is initially at rest (that is, $v = 0$ at $t = 0$) whose weight is $m = 100$ kg. Assume $c = \frac{98}{250}$ kg/m for the way this body falls.

a. Determine the time t as a function of the velocity v. Hint: you will need partial fractions.

b. Invert the function you obtained in part (a). That is, determine v as a function of t. (Note that since the body is falling, v is negative when $t > 0$.)

c. As $t \to \infty$, $v(t)$ will approach a *terminal velocity*. For this problem, what is the body's terminal velocity?

25) **Reaction Rate Expressions** If you take an introductory chemistry class, you will discuss three kinetic mechanisms by which chemical reactions occur. The models are called 0^{th}, 1^{st}, and 2^{nd} order reactions. These numbers (0, 1, and 2) correspond to the number of molecules that must collide for a reaction to occur and also to the exponent in the differential equation describing the history of the reactant chemical's concentration:

$$\begin{aligned} \frac{dc}{dt} &= -k && 0^{\text{th}} \text{ order reaction} \\ \frac{dc}{dt} &= -kc && 1^{\text{st}} \text{ order reaction} \\ \frac{dc}{dt} &= -kc^2 && 2^{\text{nd}} \text{ order reaction} \end{aligned}$$

where $k > 0$ is called the "rate constant" and $c(t)$ is the concentration of the reactant chemical. Assume that for each of these three models, the concentration at $t = 0$ is some given number c_0.

a. Use the definite integration method to show that

$$\begin{aligned} c &= c_0 - kt & \text{for a } 0^{\text{th}} \text{ order reaction,} \\ c &= c_0 e^{-kt} & \text{for a } 1^{\text{st}} \text{ order reaction, and} \\ c &= \frac{1}{\frac{1}{c_0} + kt} & \text{for a } 2^{\text{nd}} \text{ order reaction.} \end{aligned}$$

b. Assuming that concentration is in units of mol/l (that is, moles per liter) and time is in sec (that is, seconds), what are the units for k in a 0^{th} order reaction? a 1^{st} order reaction? and a 2^{nd} order reaction? Note that whether you solve this by using the solutions to the differential equations in part **a** or by using the differential equations themselves, you will get the same answer. You might want to try it both ways to check yourself.

26) **Limited population growth** In section 7.2, you learned that $y(t)$, the number of bacteria in a petrie dish, is governed by the equation $\frac{dy}{dt} = ky$, and therefore the number grows exponentially. This is true as long as the number of bacteria is small, but as the number grows, one must take into account the effect of factors that inhibit unbounded growth (like the fact that the amount of food available is finite). These are taken into account in the *logistic differential equation:*

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right).$$

Assume we have a petrie dish where $k = K = 1$ and $y(t)$ represents the number of bacteria (in 1000s). If we start with 500 bacteria (so $y(0) = 0.5$), determine $y(t)$. Hint: you will need to use partial fractions. Fun fact: If your solution is correct, you should see that $y(t) \to 1$ as $t \to \infty$, and, more generally, $y(t) \to K$ as $t \to \infty$.

27) **Chemical Equilibrium** The nature of the equilibrium between two (or more) chemical solutions is given by K, the "equilibrium constant". Although K is called a constant, it actually changes values when the temperature, T, changes via the equation

$$\frac{1}{K}\frac{dK}{dT} = \frac{\Delta H}{RT}$$

where ΔH is the difference in enthalpies between the products and reactants and R is the ideal gas law constant. If $\frac{\Delta H}{R} = 3$ and the equilibrium constant is 5 when $T = 100$ kelvin, what is it when $T = 200$ kelvin? (Note that equilibrium constants have units of concentration to some integer power. Here, we have a reaction where that integer is zero, so K is unitless.)

28) **Non-ideal Gases** In thermodynamics, it can be shown that an adiabatic process (i.e., one with no heat transfer) performed on an ideal gas will conform to the relation

$$\frac{dT}{T} = (1-\gamma)\frac{dV}{V}$$

where V is the volume of the gas, T is the temperature of the gas, and γ is a ratio between heat capacities. In practice, γ is mildly sensitive to temperature changes. If, when temperature is in kelvins,

$$\gamma = 1 + \frac{T^{0.2}}{1000}$$

and $V = 300$ liters when $T = 400$ kelvins, find V as a function of T.

29) **Gas-liquid Equilibrium** When a liquid phase and a gas phase of a substance (e.g. water and water vapor) both exist and are in equilibrium, the pressure of the gas phase, P, is related to the temperature, T. If the volume of the liquid is small compared to the gas and the gas is ideal, this relationship is given by the differential equation

$$\frac{dP}{dT} = \frac{PH^{lv}}{RT^2}$$

where R is the ideal gas law constant and H^{lv} stays essentially constant. If we use units where $R = 8$, $H^{lv} = 32$, and, at $T = 100$, $P = 3$, determine the pressure, P, at an arbitrary temperature, T.

30) **Electrical Circuit** The capacitor charge, q, in a circuit with a resistor and a capacitor is described by

$$\varepsilon = R\frac{dq}{dt} + \frac{q}{C}$$

where ε, the electromotive force, R, the resistance, and C, the capacitance, are all constants. If $\varepsilon = 2$ volts, $R = 1$ ohm, $C = 1$ farad, and t is measured in seconds, determine $q(t)$ given that $q(0) = 0$. Note that since are using SI units here, $q(t)$ will have units of coulombs.

31) **Heating or Cooling by Convection** If air at a constant temperature, T_c, flows over a small solid object whose temperature is T, the object's temperature is determined by Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - T_c)$$

where t is time and k is a experimentally determined constant with units of reciprocal time (so if t is in sec (seconds), k is in $\frac{1}{\text{sec}}$). If the temperature, T, equals some value $T_0 > T_c$ at $t = 0$, determine $T(t)$. Note: if your answer is correct, you will able to verify that $T \to T_c$ as $t \to \infty$.

Problems 32–34 are adapted from undergraduate courses and texts in economics and finance. Problems 32 and 33 involve solving separable differential equations. Problem 34 involves solving a linear differential equation. For all three problems, time, t, is in years and money, y, is in dollars.

32) **Better Interest Rates for Big Investors** In section 7.2, you learned that the value of an investment, $y(t)$, which is compounded continuously at the interest rate r, is governed by the equation $\frac{dy}{dt} = ry$. If the interest rate is related to the size of the investment by $r = \frac{\sqrt{y}}{60}$ then we have the equation

$$\frac{dy}{dt} = \frac{y^{\frac{3}{2}}}{60}.$$

a. If you initially invest \$400 (i.e., $y(0) = 400$), what is $y(t)$, the value of your investment at later times?

b. For a given initial investment, y_0, determine the time at which the worth of the investment approaches infinity (!).

33) **Funding Your Retirement** If you put money into a bank and leave it alone, its growth is governed by $\frac{dy}{dt} = ry$, as you saw in section 7.2. But what if you also add money into your bank account at a continuous rate of k dollars per year? Then the growth is governed by

$$\frac{dy}{dt} = ry + k.$$

If you remove money at k dollars per year, instead of adding it then, unsurprisingly, the equation becomes

$$\frac{dy}{dt} = ry - k.$$

Assume the bank gives a continuously compounded interest rate of 5%, so $r = 0.05$. Let's say you start working at age 20 with no money in your retirement account at the bank, but for the next 45 years you put money into your retirement fund at a constant rate of $5000 per year.

a. When you retire at age 65, how much money will you have?

b. At age 65 you stop putting money into your account. Instead you draw it out at a constant rate of, say, $60,000 per year. How old will you be when you run out of money?

c. Redo parts **a** and **b** if you start adding money at the rate of $5000 per year beginning at age 30, instead of age 20, and still retire at age 65. The results are very different!

34) **Linear Differential Equations in Retirement Savings** In the previous problem, we saw that the separable differential equation

$$\frac{dy}{dt} = ry + k$$

describes the increase in a retirement account's worth if we save at a constant rate of k dollars per year. But we normally make more money over time and can therefore save at a higher rate as we age, so a more realistic model is to have $k = a + bt$, where a is the (positive) initial contribution rate at age 20, b is the (positive) rate at which the contribution rate increases over time, and t is the time after age 20 (so $t = 0$ at age 20, and $t = 45$ at age 65). This yields the differential equation

$$\frac{dy}{dt} = ry + a + bt,$$

which is *not* separable. It is, however, a linear differential equation, which you will study more if you take Math 22 or Applied Math 106. To solve this linear equation, we subtract ry from both sides and then multiply by the integrating factor e^{-rt}, which yields

$$e^{-rt}\frac{dy}{dt} + e^{-rt}(-r)y = e^{-rt}(a + bt).$$

By the product rule, the left-hand side can be rewritten as $\dfrac{d(ye^{-rt})}{dt}$. Using this, we can now solve our equation by integrating both sides with respect to t.

Use this method to determine the worth of your retirement account if you start putting in money at age 20 and stop at age 65. Use $r = 0.05, a = 5000$, and $b = 100$, which means you are contributing an additional \$100 dollars more every year compared to the scenario in part **a** of problem #33.

Answers to odd problems

1. $y = \left(\frac{3}{2}x^2 + C\right)^{\frac{1}{3}}$
3. Not separable
5. $y = \tan^{-1}\left(\ln(|x+1|) + C\right)$
7. $f = \left(Ce^{e^{-t}(t+1)} - 1\right)^{-1}$
9. $\frac{1}{y} = \frac{3}{2} - t$
11. $\sin(3y) = \frac{3}{2}(\cos(2x) + 1)$
13. $\frac{1}{2}\left(1 - \frac{1}{c^2}\right) = t(\ln(t) - 1) - e^2$
15. $\ln\left(\frac{M-3}{M-2}\right) = x - 1 + \ln\left(\frac{1}{2}\right)$
17. $\ln\left(\frac{13(f-1)}{12f}\right) = -e^{-t}(1+t) + 1$
19. $\sin(h) - \frac{1}{3}\sin^3(h) = \frac{e^p}{2}(\sin(p) - \cos(p)) + \frac{1}{2}$
21. $y = 2e^t + 1$
23. $v = 10(\sqrt{5} - 2)\frac{\text{km}}{\text{sec}}$
25. **b.** 0^{th} order: $\frac{\text{mol}}{\text{l}\cdot\text{sec}}$, 1^{st} order: $\frac{1}{\text{sec}}$, 2^{nd} order: $\frac{\text{l}}{\text{mol}\cdot\text{sec}}$.
27. $K = 40$
29. $P = 3e^{4\left(\frac{1}{100} - \frac{1}{T}\right)}$
31. $T = T_c + (T_0 - T_c)e^{-kt}$
33. **a.** $\$100,000\left(e^{\frac{9}{4}} - 1\right) = \$848,773.58$, **b.** 89.57 years old, **c.** $\$100,000\left(e^{\frac{7}{4}} - 1\right) =$ $\$475,460.27$, 75.09 years old