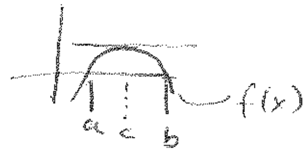
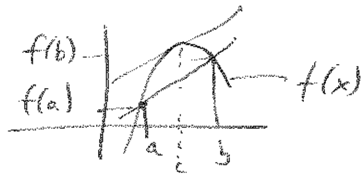
4.2 <u>Rolle's Thm</u>



If $f(a) = f(b) = 0$

there is at least one pt c, $a < c < b$, s.t. $f'(c) = 0$.

<u>Mean Value Thm</u>



Given a and b,

slope of line through

$(a, f(a))$ and $(b, f(b))$ is $m = \dfrac{f(b)-f(a)}{b-a} = \dfrac{\Delta y}{\Delta x}$

MVThm says there is at least one pt c, $a < c < b$,

s.t. $f'(c) = m = \dfrac{f(b)-f(a)}{b-a}$

or $f(b) - f(a) = f'(c)(b-a)$

EX p 235 #7 $\quad f(x) = x^3 - x^2 \quad [-1, 2]$

$\Rightarrow f'(x) = 3x^2 - 2x \quad \therefore\ 3c^2 - 2c = c(3c-2)$.

$$m = \frac{(8-4)-(-1-1)}{2-(-1)} = \frac{4-(-2)}{2+1} = \frac{6}{3} = 2$$

$$3c^2 - 2c - 2 = 0$$

$$\Rightarrow c = \frac{2 \pm \sqrt{4-4 \cdot 3(-2)}}{6} = \frac{2 \pm \sqrt{4+24}}{6} = \frac{2 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{6}$$

$$= \frac{1 \pm \sqrt{7}}{3} \doteq 1.22,\ -0.549 \quad \text{(both in interval)}$$

$$f'\left(\frac{1+\sqrt{7}}{3}\right) = \cancel{3}\left(\frac{1+2\sqrt{7}+7}{\cancel{9}_3}\right) - \frac{2+2\sqrt{7}}{3} = \frac{6}{3} = \underline{2}$$

$$f'\left(\frac{1-\sqrt{7}}{3}\right) = \cancel{3}\left(\frac{1-2\sqrt{7}+7}{\cancel{9}_3}\right) - \frac{2-2\sqrt{7}}{3} = \frac{6}{3} = \underline{2}$$

$\doteq m$

4.3 – 4.4 1st and 2nd Derivative Tests

Given $y = f(x)$, if $f'(x) > 0$ for some x, f is rising
and if $f'(x) < 0$ for some x, f is falling

If $f'(x) = 0$ for some x, this is a "critical" point and
The slope is flat. Possibly a local max or min.

Concavity

If a section of $f(x)$ opens downward, 
it is said to be concave DOWNWARD.

If a section of $f(x)$ opens upward, ,
it is said to be concave UPWARD.

2nd Deriv and Concavity

If $f''(x) > 0$ for some x, the curve is concave upward at x
If $f''(x) < 0$ for some x, the curve is concave downward at x

Eg. $y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2 > 0$

 ← concave upward.

If $f''(x) = 0$, x is a pt of inflection, where the concavity changes.

4.5 Indeterminate Forms and L'Hôpital's Rule

Def If f and g are cont's at $x=a$ and $f(a)=g(a)=0$,

then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called and indeterminate form

An indet. form cannot be evaluated directly.

One can try algebra. EG.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim \frac{(x+1)(x-1)}{x-1} = \lim x+1 = 2$$

L'Hôpital's Rule If $f(x_0)=g(x_0)=0$ and both f, g are diff'ble (except possibly at x_0)

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ Provided $\frac{f'(x)}{g'(x)}$ has a limit as $x\to x_0$

EG

$$\lim_{x\to 0} \frac{\sin x}{x} \left(\frac{0}{0}\right) \overset{L'H}{=} \lim \frac{\cos x}{1} = \frac{1}{1} = 1$$

also works for $f(x_0)=g(x_0)\to\infty$ and $\frac{f(x_0)}{g(x_0)}$

or $f(x_0) - g(x_0)$

EG.
$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim \frac{x-\sin x}{x\sin x} \overset{L'H}{=} \lim \frac{1-\cos x}{\sin x + x\cos x}$$

$$\overset{L'H}{=} \lim \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$$

4.8 Anti derivatives

An anti-derivative or indefinite integral is the reverse of differentiation.

Eg. $\frac{d}{dx}(x^2) = 2x \Rightarrow \int 2x\,dx = x^2 + C$

$\Rightarrow$ Need to remember basic derivative rules!

$$\int dx = x + C$$

$$\int a f(x)\,dx = a \int f(x)\,dx \quad (\underline{a} \text{ a constant})$$

$$\int u(x) + v(x)\,dx = \int u(x)\,dx + \int v(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = +\sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{dx}{1+x^2} = \arctan x + C$$

etc.

Solving diff. eq.

$$\frac{dy}{dx} = \frac{1}{x^2} + x \quad \text{given } y(2)=1$$

$$\Rightarrow \int dy = \int x^{-2} + x \, dx$$

$$\Rightarrow y = \frac{x^{-1}}{-1} + \frac{x^2}{2} + C = -\frac{1}{x} + \frac{x^2}{2} + C$$

Subst. $(2,1)$

$$\Rightarrow 1 = \frac{-1}{2} + \frac{2^2}{2} + C = -\frac{1}{2} + 2 + C = \frac{3}{2} + C$$

$$\Rightarrow 1 - 3/2 = C = -\frac{1}{2}$$

$\therefore$ sol of diff. eq.

$$y = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}$$

5.5 Substitution

① $\int x(3x^2+2)^3\,dx$

$u = 3x^2+2$

$\frac{du}{dx} = 6x$

$\frac{du}{6} = x\,dx$

$= \frac{1}{6}\int u^3\,du$

$= \frac{1}{6}\frac{u^4}{4} + C = \frac{1}{24}(3x^2+2)^4 + C$

② $\int x\cos x^2\,dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

$= \frac{1}{2}\int \cos u\,du$

$= \frac{1}{2}\sin u + C = \frac{\sin x^2}{2} + C$

③ $\int \sin^2 x\cos x\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

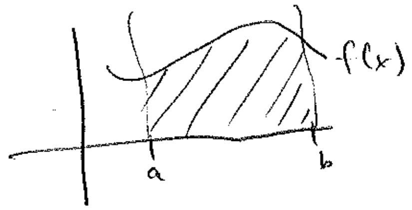
$du = \cos x\,dx$

$= \int u^2\,du$

$= \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$

5.3 **Definite Integral**

Given $f(x)$ between $x=a, b$, we indicate the area under the curve as $A = \int_a^b f(x)\,dx$



Approximating the area by rectangles + taking the limit as the number of rectangles increases, we see that A is

$$A = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \lim \sum \text{height} \cdot \text{width}.$$

Properties

Since $\int_a^b f(x)\,dx$ is defined in terms of sums, the following follow:

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

etc.

5.4 Fundamental Thm of Integral Calculus

Thm If f is cont's on $[a,b]$ and F is s.t.

$$\frac{dF}{dx} = f \quad \text{or} \quad \int f(x)\,dx = F(x) + C$$

THEN $$\int_a^b f(x)\,dx = F(b) - F(a).$$

Ex

① $$\int_1^2 x^3\,dx = \frac{x^4}{4}\Big]_1^2 = \frac{2^4}{4} - \frac{1}{4} = \frac{16-1}{4} = \frac{15}{4}$$

② $$\int_0^4 \sqrt{2x+1}\,dx \quad \begin{bmatrix} u = 2x+1 \\ \frac{du}{dx} = 2 \\ \frac{du}{2} = dx \end{bmatrix} = \frac{1}{2}\int_{x=0}^4 u^{1/2}\,du = \frac{1}{\cancel{2}}\frac{u^{3/2}}{3/\cancel{2}}\Big]_{x=0}^4$$

$$= \frac{(2x+1)^{3/2}}{3}\Big]_0^4 = \frac{9^{3/2}}{3} - \frac{1^{3/2}}{3} = \frac{27-1}{3} = \frac{26}{3}$$

③ (5.5) area between 2 curves

$$\begin{aligned} y_1 &= 2x - x^2 \\ y_2 &= -3 \end{aligned} \quad 2x - x^2 = -3 \Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3, -1$$

y_1 is upper curve since $y_1 = 0$ at 0 and $y_2 = -3$ at 0

$$\int_{-1}^3 2x - x^2 - (-3)\,dx = \frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3} + 3x\Big]_{-1}^3$$

$$= 9 - \frac{27}{3} + 9 - \left((-1)^2 - \frac{(-1)^3}{3} - 3\right)$$

$$= 9 - 1 - \frac{1}{3} + 3 = 11 - \frac{1}{3} = 10\tfrac{2}{3}$$

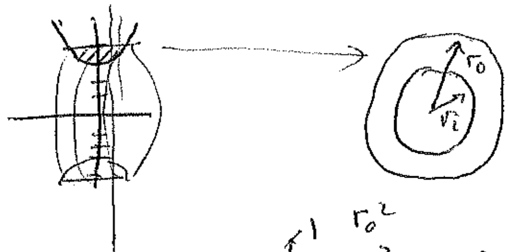
6.1 Volumes

Slices (Cross sections) $\quad V = \pi \int r^2 \, dx$ or $\pi \int r_o^2 - r_i^2 \, dx$

solid $\qquad$ hollow

Rotate area between $y = 3 + x^2$ and $y = 4$ around x-axis and find volume.

$4 = 3 + x^2$
$\Rightarrow 1 = x^2$
$\Rightarrow x = \pm 1$



$r_o = 4$
$r_i = 3 + x^2$

$$V = \pi \int_{-1}^{1} \overbrace{4^2}^{r_o^2} - \overbrace{(3+x^2)^2}^{r_i^2} \, dx$$

$$= \pi \int_{-1}^{1} 16 - (9 + 6x^2 + x^4) \, dx$$

$$= \pi \int 7 - 6x^2 - x^4 \, dx$$

$$= \pi \left[7x - \frac{\overset{2}{\cancel{6}}x^3}{\cancel{3}} - \frac{x^5}{5} \right]_{-1}^{1}$$

$$= \pi \left[\left(7 \cdot 1 - 2 \cdot 1 - \frac{1}{5}\right) - \left(7(-1) - 2(-1) - \frac{(-1)}{5}\right) \right]$$

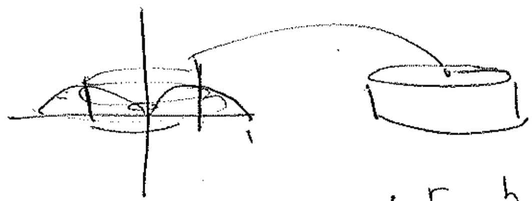
$$= \pi \left[7 - 2 - \frac{1}{5} + 7 - 2 - \frac{1}{5} \right]$$

$$= \pi \left[10 - \frac{2}{5} \right] = \underline{\frac{48\pi}{5}}$$

6.2 Cylinders / Shells $\qquad V = \int 2\pi r h \, dx$

Rotate area between $y = x - x^2$ and x-axis around y-axis and find volume.

$$x - x^2 = 0$$
$$x(1-x) = 0$$
$$\hookrightarrow x = 0, 1$$



$r = x$
$h = x - x^2$

$$V = 2\pi \int_0^1 \overset{r}{x}\,\overset{h}{(x - x^2)}\,dx$$
$$= 2\pi \int x^2 - x^3 \, dx$$
$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$
$$= 2\pi \left[\left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0) \right]$$
$$= \cancel{2}\pi \left(\frac{4-3}{\cancel{12}_6} \right) = \underline{\frac{\pi}{6}}$$

6.3 Length along curve

Formulas derived from Pythagorean Thm – calculating length of hypoteneuse.

$$L = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex $\quad x = \frac{t^2}{2} \quad y = \frac{1}{3}(2t+1)^{3/2} \quad t = 0 \to 4$

$$\frac{dx}{dt} = \frac{\cancel{2}t}{\cancel{2}} = t \quad \frac{dy}{dt} = \frac{1}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(2t+1)^{1/2} \cdot \cancel{2} = (2t+1)^{1/2}$$

$$L = \int_0^4 \sqrt{(t)^2 + \left[(2t+1)^{1/2}\right]^2}\, dt = \int \sqrt{t^2 + 2t + 1}\, dt = \int \sqrt{(t+1)^2}\, dt$$

$$= \int t+1\, dt = \frac{t^2}{2} + t \Big]_0^4 = \frac{16}{2} + 4 - 0 = \underline{12}$$

7.1 <u>Exponential e and Natural Log</u>

$e = 2.7182818\ldots$

$\log_e x = \ln x = \int_1^x \frac{1}{t}\,dt$

$\frac{d}{dx}(\ln x) = \frac{1}{x} \Leftrightarrow \int \frac{1}{x}\,dx = \ln|x| + C$

EX ① $\int \frac{x^2}{4-x^3}\,dx$ $\left[\begin{array}{l} u = 4-x^3 \\ \frac{du}{dx} = -3x^2 \\ \frac{du}{-3} = x^2\,dx \end{array}\right]$ $= -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|4-x^3| + C$

② $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\left[\begin{array}{l} u = \cos x \\ \frac{du}{dx} = -\sin x \\ -du = \sin x\,dx \end{array}\right]$

$= -\int \frac{1}{u}\,du$

$= -\ln|u| + C = \underline{-\ln|\cos x| + C}$

③ $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$ $\left[\begin{array}{l} u = 1 + x^{1/2} \\ \frac{du}{dx} = \frac{1}{2}x^{-1/2} \\ 2du = \frac{dx}{x^{1/2}} \end{array}\right]$ $= 2\int \frac{du}{u}$

$= 2\ln|u| + C = \underline{2\ln|1+\sqrt{x}| + C}$

8.1 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Ex $\int x\,e^x\,dx$ $\quad u = x \quad dv = e^x\,dx$

$\quad du = dx \quad v = e^x$

$$= x e^x - \int e^x\,dx = \underline{x e^x - e^x + C}$$

Ex $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$ $\quad u = \sec x \quad dv = \sec^2 x\,dx$

$\quad du = \sec x \tan x\,dx \quad v = \tan x$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow \quad 2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\Rightarrow \quad \int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C$$

8.3 Trig Integrals

EX 1

$$\int \sin^2 x\,dx = \frac{1}{2}\int 1-\cos 2x\,dx$$
$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$$
$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

EX 2

$$\int \sin^3 x\,dx = \int \sin x\,(\sin^2 x)\,dx$$
$$= \int \sin x\,(1-\cos^2 x)\,dx$$
$$= \int \sin x\,dx - \int \cos^2 x \sin x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\cos x + \int u^2\,du$$
$$= -\cos x + \frac{u^3}{3} + C$$
$$= -\cos x + \frac{\cos^3 x}{3} + C$$

EX 3

$$\int \sin^2 x \cos^2 x\,dx = \int \sin^2 x\,(1-\sin^2 x)\,dx$$
$$= \int \sin^2 x - \sin^4 x\,dx$$
$$= \int \frac{1}{2}(1-\cos 2x) - \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx$$
$$= \int \frac{1}{2} - \frac{\cancel{\cos 2x}}{\cancel{2}} - \frac{1}{4} + \frac{\cancel{2}\cancel{\cos 2x}}{\cancel{4}\,2} - \frac{\cos^2 2x}{4}\,dx$$
$$= \frac{1}{4}\int 1 - \cos^2 2x\,dx$$
$$= \frac{1}{4}\int dx - \frac{1}{4}\int \cos^2 2x\,dx$$
$$= \frac{x}{4} - \frac{1}{8}\int 1 + \cos 4x\,dx$$
$$= \frac{x}{4} - \frac{x}{8} - \frac{\sin 4x}{32} + C$$
$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

8.4 Trig Substitutions

Given expression	use
$1+u^2$	$u=\tan x$
$1-u^2$	$u=\sin x$
u^2-1	$u=\sec x$

Eb. $\int \sqrt{1+u^2}\,du$ (can't use u-sub since no "du")

$u=\tan x \qquad du=\sec^2 x\,dx$

$1+u^2 = 1+\tan^2 x = \sec^2 x$

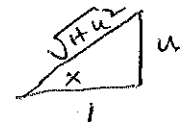
$= \int \sqrt{\sec^2 x}\,\sec^2 x\,dx$

$= \int \sec^3 x\,dx$

using integ by parts.

$= \dfrac{\sec x\tan x}{2} + \dfrac{1}{2}\ln|\sec x+\tan x) + C$

$u=\tan x \Rightarrow$



$\Rightarrow \sec x = \sqrt{1+u^2}$

$= \dfrac{u\sqrt{1+u^2}}{2} + \dfrac{1}{2}\ln\left|\sqrt{1+u^2}+u\right| + C$

→ May need to "complete the square" first

$\int \dfrac{x\,dx}{\sqrt{x^2-2x+5}}$

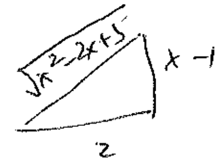
$x^2-2x+5 = x^2-2x+1+4$
$= (x-1)^2+4$

$= \int \dfrac{x\,dx}{\sqrt{(x-1)^2+4}}$

$= \dfrac{1}{2}\int \dfrac{x\,dx}{\sqrt{\left(\frac{x-1}{2}\right)^2+1}}$

$\dfrac{x-1}{2} = \tan\theta \Rightarrow x = 1+2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$

$= \dfrac{2}{2}\int \dfrac{(1+2\tan\theta)\cancel{\sec^2}\theta\,d\theta}{\cancel{\sec\theta}}$



$= \int \sec\theta + 2\sec\theta\tan\theta\,d\theta$

$\sec\theta = \dfrac{\sqrt{x^2-2x+5}}{2}$

$= \ln|\sec\theta+\tan\theta| + 2\sec\theta + C$

8.5 Partial Fractions

$$\int \frac{x\,dx}{x^2-2x-3}$$

$$x^2-2x-3 = (x-3)(x+1)$$

$$\therefore \frac{x}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$\begin{aligned} x &= A(x+1) + B(x-3) \\ &= Ax + A + Bx - 3B \\ &= (A+B)x + A - 3B \end{aligned}$$

$$\Rightarrow \begin{aligned} 1 &= A+B \quad (1) \\ 0 &= A-3B \quad (2) \end{aligned}$$

$$(1)-(2):\ 1 = 4B \Rightarrow B = 1/4$$

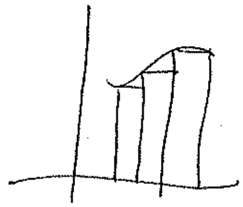
$$A = 3/4$$

$$\therefore \int \frac{x\,dx}{x^2-2x-3} = \frac{3}{4}\int \frac{dx}{x-3} + \frac{1}{4}\int \frac{dx}{x+1}$$

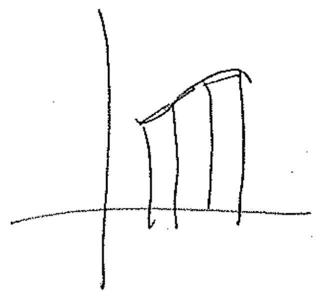
$$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C$$

8.7 Numerical Integration

Rectangle Rule

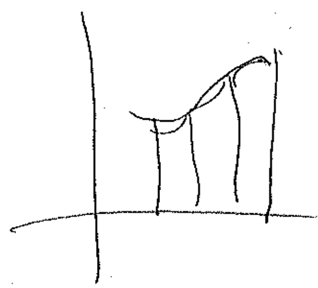


Trapezoidal Rule



error related to f''

Simpson's Rule
(use parabolas)



error related to $f^{(iv)}$

8.8 Improper Integrals

If a function goes to infinity in the range of integration or the interval of integration is infinite, the integral is said to be "improper". Usually, a limit approach is used to evaluate.

① $\int_1^\infty \frac{1}{x^2}\,dx \to \lim_{b\to\infty} \int_1^b x^{-2}\,dx$

$= \lim \left[\frac{x^{-1}}{-1}\right]_1^b = \lim -\frac{1}{b} - (-1)$

$= \lim_{b\to\infty} 1 - \frac{1}{b} \nearrow 0 = 1$

② $\int_1^\infty \frac{1}{x}\,dx \to \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim \left[\ln|x|\right]_1^b = \lim(\ln b - \ln 1 \nearrow 0) = \lim_{b\to\infty} \ln b = \infty$

③ $\int_{-1}^1 \frac{1}{x^2}\,dx \to \lim_{b_1\to 0^-} \int_{-1}^{b_1} x^{-2}\,dx$ and $\lim_{b_2\to 0^+} \int_{b_2}^1 x^{-2}\,dx$

$\to \lim \left[\frac{x^{-1}}{-1}\right]_{-1}^{b_1} = \lim \left[-\frac{1}{b} + (-1)\right]$

$= \lim \left[-1 - \frac{1}{b}\right] = \infty$