(87-6)

PRELIM

Indefinite integrals are the reverse of differentiation defined as the slope of a curve (using limits)

Definite integrals are defined as the area under the graph of a curve using the limit of a sum of rectangles.

It would be nice if we could relate the 2 concepts since then we could avoid the mess associated with finding limits when we are looking for area under a curve.

This is the importance of the next theorem.

(5.4) ~~(6.7)~~ p 325

The FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Thm ~~#~~ (p: ~~282~~) 328

Let f be continuous on $[a,b]$

Let F be such that $\frac{dF}{dx} = f$

i.e. F is the antiderivative of f

i.e. $\int f(x)\,dx = F(x) + C$.

THEN

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

i.e. $\lim_{n\to\infty} \sum f(c_i)\Delta x_i$ = area under $f(x)$ between a and b

$= F(b) - F(a)$ = difference of indefinite integrals (antiderivatives) evaluated at b and a.

NOTE: This theorem interconnects the concept of <u>definite integrals</u> (defined as a limit of a sum and related to the area under a curve) with <u>indefinite integrals</u> (defined as the reverse process of differentiation)!

EXAMPLES

~~EX 5 283~~ cf ~~EX 2~~ prev. sect. ~~(p 275)~~

$$\int_0^b x^2\,dx = \frac{x^3}{3}\Bigg]_0^b = \frac{b^3}{3} - \frac{0^3}{3} = \frac{b^3}{3} \quad //$$

~~EX 3~~ ~~p 286 #18a~~

Calculate area between x-axis + parabola $y = 6 - x - x^2$

Step 1 Find pts where parabola crosses axis. $\Rightarrow$ set $y = 0$

$$0 = 6 - x - x^2 = (3+x)(2-x)$$

$$\Rightarrow x = 2, -3$$

Step 2 Evaluate

$$\int_{-3}^{2} 6 - x - x^2\,dx = 6x - \frac{x^2}{2} - \frac{x^3}{3}\Bigg]_{-3}^{2}$$

$$= \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right)$$

$$= 10 - \frac{8}{3} + 18 + \frac{9}{2} - 9$$

$$= 19 - \frac{16}{6} + \frac{27}{6} = 19 + \frac{11}{6} = \underline{\underline{20\tfrac{5}{6}}}.$$

NEGATIVE FUNCTIONS

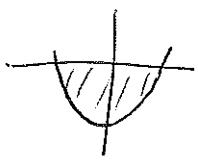
- p 331-2.

If a function is negative over an interval of integration, the numerical result from the definite integral will be negative and is, therefore, the negative of the area.

~~EX 8~~ ~~p 283~~ 12

12 cf EX 6 p 331

Find area between $y = x^2 - 4$ and x-axis between $x = -2$ to 2.



$$\int_{-2}^{2} x^2 - 4 \, dx = \frac{x^3}{3} - 4x \Big]_{-2}^{2}$$

$$= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)$$

$$= \frac{16}{3} - 16 = \frac{16 - 48}{3} = \frac{-32}{3}$$

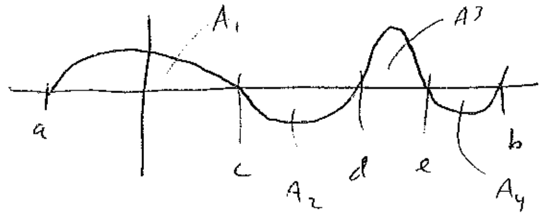
$\therefore$ Area is $+\underline{\underline{\frac{32}{3}}}$.

<u>Note:</u> One can also envision this as reversing the curve (mirror image about x-axis), thereby getting a positive function, and then getting the area of the new curve (which is identical).

ALTERNATING FUNCTIONS (pg ~~289~~) 332 – 333

What if the given curve is positive and negative?
i.e. goes up + down

eg.



Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^e f(x)\,dx + \int_e^b f(x)\,dx$$

$$= A_1 \quad - A_2 \quad + A_3 \quad - A_4$$

i.e we get the algebraic sum of "signed" areas.

$\Rightarrow$ If we want total actual area, we must calculate each area separately and add absolute values.

q EX8 p 333

EX~~9~~ ~~(cf p. 275)~~ Find total area bounded by curve $y = x^3 - 4x$ and x-axis.

① Find points of intersection i.e. set $y=0$

$\therefore\ x^3 - 4x = 0 \Rightarrow 0 = x(x^2-4) = x(x+2)(x-2)$

$\therefore$ pts of intersection are $x = 0, 2, -2$.

② Set up 2 integrals.

$A_1 = \int_{-2}^{0} x^3 - 4x\,dx \qquad$ and $A_2 = \int_0^2 x^3 - 4x\,dx$

$$\left.\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{0} = \left(\frac{0}{4} - \frac{0}{2}\right) - \left(\frac{16}{4} - \frac{4\cdot 4}{2}\right) = -(4-8) = 4 \quad = A_1$$

$$\left.\frac{x^4}{4} - \frac{4x^2}{2}\right]_{0}^{2} = \left(\frac{16}{4} - \frac{4\cdot 4}{2}\right) - 0 = -4 = A_2$$

③ total area $= |A_1| + |A_2| = 4 + 4 = 8$

IMPLICATIONS

Thm ~~1 (p 279)~~ ~~28~~ R 4 p 322.

Let f be continuous on $[a,b]$ and

let $F(x) = \int_a^x f(t)\,dt$

Then $F(x)$ is differentiable at every $x \in [a,b]$

and $$\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

EX pg ~~286~~ 334 . #45 ~~7B~~ 12

Find $F'(x)$ if $F(x) = \int_0^x \sqrt{1+t^2}\,dt$

Now $f(t) = \sqrt{1+t^2}$

$\therefore$ By Thm 4, $F'(x) = f(x) = \sqrt{1+x^2}$ //

NOTE:1. If the limit is not a simple x, one must use the chain rule.

EX 2c pg ~~281~~ 327

Find $\frac{dy}{dx}$ if $y = \int_0^{x^2} \cos t\,dt$

Let $u = x^2$, then $f(u) = \int_0^u \cos t\,dt$

$\therefore \frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$\therefore \frac{dy}{dx} = (\cos u)(2x) = 2x\cos x^2$ //

LOCATION and VELOCITY

~~p 306~~ p 330

Velocity is the rate of change of the location wrt time,

i.e. $v = \frac{ds}{dt}$.

If we are given a velocity formula, we can find a formula for the location or position.

i.e

$$v = \frac{ds}{dt} = f(t)$$

$$ds = f(t)\,dt$$

$$\int ds = \int f(t)\,dt$$

$$s = \int f(t)\,dt$$

This formula will be in terms of t, if the velocity formula was in terms of t.

5.1 cf p 362

CHANGE OF POSITION / LOCATION

If we start in location a and move to location b, there are 2 major ways of describing the change.

① NET CHANGE IN POSITION or DISPLACEMENT is the (signed) distance between the locations, eg. the displacement between LA + SF is 450 miles

② TOTAL DISTANCE TRAVELED is the (signed) distance which takes into account side-trips + backward motion, eg. the total distance traveled between LA + SF by way of SD is 750 miles.

DISPLACEMENT

Net change = displacement = $\int_a^b v(t)\,dt = s(b) - s(a)$

Eg. If the location formula is $s = 3t^2 + 2t + 5$

at $t=0$, $s=5$

at $t=2$, $s = 12 + 4 + 5 = 21$

$\therefore$ the displacement in traveling between $t=0$ and $t=2$.

is $21 - 5 = \underline{\underline{16}}$ (i.e. $s(2) - s(0)$).

TOTAL DISTANCE

Total distance $= \int_a^b |v(t)|\,dt$

In effect, this means finding the intermediate points c_i's

when $v(c_i) = 0$ and finding

$$\sum \left| \int_{c_i}^{c_{i+1}} v(t)\,dt \right| + \left| \int_a^{c_1} v(t)\,dt \right| + \left| \int_{c_k}^{b} v(t)\,dt \right|$$

C EXAMPLES of EX 5 p 330

4

Pg #11 Ex $v = 2t+1 \qquad 0 \le t \le 2$

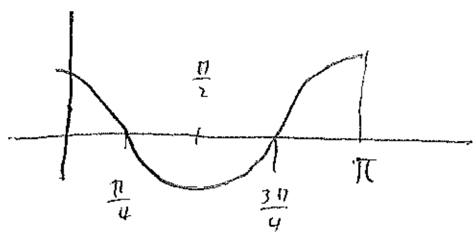
c) in problem p 308

displacement $= (\text{Tot dist. travel.}) = \int_0^2 2t+1 \, dt$

$$= \cancel{2}\frac{t^2}{\cancel{2}} + t \Big]_0^2 = 4 + 2 - 0 = \underline{\underline{6}}$$

TB p 308 #6 8

$v = 4\cos 2t \qquad 0 \le t \le \pi$



Now $\cos 2t = 0$ at $2t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

$\therefore$ when $t = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

$$\text{Tot. dist. travel} = \int_0^{\frac{\pi}{4}} 4\cos 2t \, dt - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 4\cos 2t \, dt + \int_{\frac{3\pi}{4}}^{\pi} 4\cos 2t \, dt.$$

Now $\int 4\cos 2t \, dt$

$u = 2t \Rightarrow \frac{du}{2} = dt$

$= 2\int \cos u \, du$

$= 2\sin u = 2\sin 2t$

$$\therefore \text{ we get Tot dist} = \Big[2\sin 2t\Big]_0^{\frac{\pi}{4}} - \Big[2\sin 2t\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \Big[2\sin 2t\Big]_{\frac{3\pi}{4}}^{\pi}$$

$$= 2\sin\frac{\pi}{2} - 2\sin 0 - 2\sin\frac{3\pi}{2} + 2\sin\frac{\pi}{2} + 2\sin 2\pi - 2\sin\frac{3\pi}{2}$$

$$= 2\cdot 1 - 2\cdot 0 - 2(-1) + 2(1) + 2\cdot 0 - 2(-1)$$

$$= 2 + 2 + 2 + 2 = \underline{\underline{8}}$$

HW p 333 #1, 2, 4, 5, 6, 10