RIEMANN INTEGRAL

- It can be shown that one can use circumscribed rectangles, (i.e. rectangles which contain the curve), take the limit and get the same answer.
- It can also be shown that the widths of the rectangles need not be the same, but in the limit, one still gets the same answer
- Since it doesn't make any difference how we set up the rectangles to approximate the area, we use one standard notation to indicate the limit of the approximating sums:

$$\int_a^b f(x)\,dx \quad \text{(called the Riemann integral)}$$

which indicates

$$A = \lim_{\max \Delta x_k \to 0} \sum f(c_k)\,\Delta x_k$$

where $f(c_k)\Delta x_k$ are rectangles associated with the curve $f(x)$ over the interval $[a, b]$.

__Thm 1__, ~~(p 267)~~ p 315

If f is continuous on $[a, b]$, then the Riemann integral

$$\int_a^b f(x)\,dx = \lim_{\substack{\cancel{\max \Delta x_k} \to 0 \\ \text{norm}}} \sum f(c_k)\,\Delta x_k$$

exists.

Moreover, no matter how one chooses the c_k in the subintervals, one gets the same number.

Def ~~Corollary~~ Informal, ~~top p 268~~ p 319 (~~skip 8b~~)

If $f(x) \geq 0$, then the area under the graph of $f(x)$ between $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.

__NOTE!__ BEWARE. Right now, $\int_a^b f(x)\,dx$ is defined as a limit of a sum. It has no relationship to the [illegible] of differentiation as yet!!!

NOTE: (1) $\int$ is a modified S [illegible] of SUM

(2) a, b are "limits of integration"

a is "lower limit"

b is "upper limit"

DEFINITE INTEGRALS (~~4.2~~6.8) p314

Def The Riemann integral $\int_a^b f(x)\,dx$

is also called the <u>definite integral of f over $[a,b]$</u>.

<u>NOTE:</u> Since this notation gives us a <u>numerical value</u>, the so-called "variable of integration" is unimportant to the extent that it can be changed if circumstances warrant it.

In other words,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du \quad \text{etc.}$$

p 315

$\therefore$ variable of integration is sometimes called the "dummy variable"

PROPERTIES OF DEF. INTEGRALS

~~(p 269 table 4.1)~~ p317 table 5.4

Because definite integrals are defined as limits of finite sums, they possess certain properties derived from the properties of limits and finite sums.

(3) I1. $$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$ for any constant k (similar to property of indef. int).

4,5 (4) I2-I3 $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$ (similar to prop. of indef. int)

(7) I4 $$\int_a^b f(x)\,dx \geq 0 \quad \text{if } f(x) \geq 0 \text{ on } [a,b]$$

5 (7) I5 $$\int_a^b f(x)\,dx \gtreqless \int_a^b g(x)\,dx \quad \text{if } f(x) \gtreqless g(x) \text{ on } [a,b]$$

(6) I6 $$(\min f)(b-a) \leq \int_a^b f(x)\,dx \leq (\max f)(b-a)$$

(1) I7 $$\int_b^a f(x)\,dx \overset{def}{=} -\int_a^b f(x)\,dx$$

(2) I8 $$\int_a^a f(x)\,dx \overset{def}{=} 0$$

(5) I9 $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

provided $f(x)$ is continuous on $[a,c]$

EX 3 ~~16~~ 15 p. 218

Given $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$

1. ~~4.~~ $\int_{-1}^{1} 3 f(x)\,dx = 3 \int_{-1}^{1} f(x)\,dx = 3(5) = 15.$

3 ~~4.~~ $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

B CALCULATING AREAS BY LIMIT DEF

(5.3) 2D

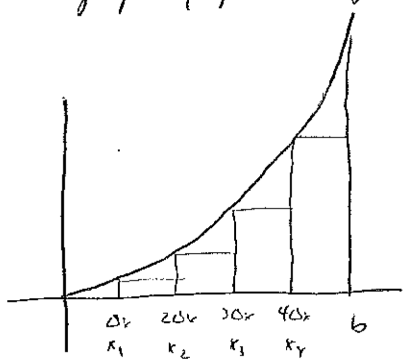
~~EX 2 pg 275~~

p 322

#65

with $a=0$

Find area under graph of $y = x^2$ for $0 \le x \le b$



We calculate $\int_0^b x^2\,dx$ from its limit definition.

Subdivide intervals into n subintervals. $\therefore\ \Delta x = \frac{b-0}{n} = \frac{b}{n}$

Areas of rectangles (inscribed)

1st. $f(0)\Delta x = 0$

2d $f(x_1)\Delta x = f(\Delta x)\Delta x = \Delta x^2 \Delta x = \Delta x^3$

3rd $f(x_2)\Delta x = f(2\Delta x)\Delta x = 2^2\Delta x^2\Delta x = 2^2\Delta x^3$

$\vdots$

last $f(x_{n-1})\Delta x = f((n-1)\Delta x)\Delta x = (n-1)^2\Delta x^2\Delta x = (n-1)^2\Delta x^3$

$$\int_0^b x^2\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x = \lim\left[\left(1^2+2^2+\dots+(n-1)^2\right)\Delta x^3\right]$$

$$= \lim\left[\frac{(n-1)\,n\,(2n-1)}{6}\ \frac{b^3}{n^3}\right]$$

$$= \lim\left[\frac{b^3}{6}\ \frac{n-1}{n}\cdot\frac{n}{n}\cdot\frac{2n-1}{n}\right]$$

$$= \lim\left[\frac{b^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

$$= \frac{b^3}{6}(1)(2) = \frac{b^3}{3}$$

if $b=8$

$\frac{512}{3} = 170.666$

NOTES

① This is a general formula

i.e. $\int_0^b x^2 dx = \frac{b^3}{3}$

$\therefore \int_0^2 x^2 dx = \frac{2^3}{3} = \frac{8}{3}$

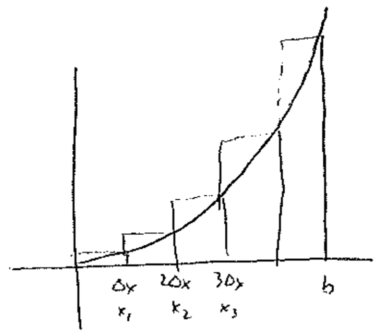
② Notice similarity to

$$\int x^2 dx = \frac{x^3}{3} + C$$

C EXAMPLE

Skip if needed

7 p276 #4 Use circumscribed rectangles to calculate area for x^2 from 0 to b.



Again subdivide intervals into n subintervals $\therefore \Delta x = \frac{b-0}{n} = \frac{b}{n}$

Areas of rectangles

$$
\begin{aligned}
&\text{1st.} && f(x_1)\Delta x = f(\Delta x)\Delta x = \Delta x^2 \Delta x = \Delta x^3 \\
&\text{2nd} && f(x_2)\Delta x = f(2\Delta x)\Delta x = 2^2\Delta x^2 \Delta x = 2^2 \Delta x^3 \\
&\text{3rd} && f(x_3)\Delta x = f(3\Delta x)\Delta x = 3^2\Delta x^2 \Delta x = 3^2 \Delta x^3 \\
& && \vdots \\
&\text{last} && f(b)\Delta x = f(b)\Delta x = (n\Delta x)^2 \Delta x = n^2 \Delta x^3
\end{aligned}
$$

$$
\int_0^b x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim \left[(1^2 + 2^2 + \dots + n^2)\Delta x^3 \right]
$$

$$
= \lim \left[\frac{n(n+1)(2n+1)}{6} \frac{b^3}{n^3} \right]
$$

$$
= \lim \left[\frac{b^3}{6} \frac{\cancel{n}}{\cancel{n}} \frac{n+1}{n} \frac{2n+1}{n} \right]
$$

$$
= \lim \left[\frac{b^3}{6} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right]
$$

$$
= \frac{b^3}{6}(1)(2) = \frac{b^3}{3} \;//
$$

Same as with inscribed rectangles.