2004 (4.3 / 4.4) 287 (93-4) 5-1 (89-2)

SUBSTITUTION (5.5) p 336

- The "guessing" method (with subsequent "correction") is (p 242) not always the most efficient method to perform integration
- The more standard method is that of "substitution".

The general rule is:

- find the more complicated expression in the integral (usually it contains a power)
- (1) substitute a new variable for the expression (WITHOUT the power).
- (2) take the derivative & (3) separate variables
- substitute for the expression + the derivative
- integrate
- re-substitute

EX ~~4~~

$$\int \frac{x\,dx}{\sqrt{4-x^2}}$$

More complicated expression is $(4-x^2)^{-\frac{1}{2}}$

$\therefore$ let (1) $u = 4-x^2$

(2) $du = -2x\,dx$

(3) $\frac{du}{-2} = x\,dx$

$$= \int u^{-\frac{1}{2}} \boxed{\frac{du}{-2}} = \frac{1}{-2}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{1/2}\right] + C$$

$$= -u^{\frac{1}{2}} + C = -\sqrt{4-x^2} + C$$

SUBSTITUTION (4.3)

Very important

→ gen principle – look for "something" and its Derivative →

TB pg 252 #18

$$\int \frac{y\,dy}{\sqrt{2y^2+1}}$$

We know

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

→ need to make substitution.

(A) – rewrite via algebra

not "plain old ord. y all by itself"

$$\int y\,(2y^2+1)^{-1/2}\,dy$$

(B) look for "something" ← usually more complicated express.

3 steps

subst. (1) $u = 2y^2+1$

diff. (2) $\frac{du}{dy} = 4y$

sep. var (3) $\frac{du}{4} = y\,dy$

$$\frac{1}{4}\int u^{-1/2}\,du$$

$= \ldots$

NOTES:

1. In many cases, more than one choice is possible for the substitution – cf EX 5 – [illegible]
2. Substitution works because of the Chain Rule (in one sense it is the backwards use of the chain rule).
3. Once in a while a second substitution is necessary.
4. In any integral, there is only 1 variable (matching final differential - variable)

   EITHER ALL old variable or all New variable

   NEVER $\frac{1}{2}$ & $\frac{1}{2}$.

(Not .. PP)

EX #: 5 7A ... of EX 8 $\quad \displaystyle\int \frac{z+1}{\sqrt[3]{3z^2+6z+5}}\,dz$

p346

<u>attempt #1</u> $\Rightarrow$ messiest expression (without power) is $3z^2+6z+5$

$$① \; u = 3z^2+6z+5$$

$$② \; du = (6z+6)\,dz = 6(z+1)\,dz$$

$$\Rightarrow ③ \; \frac{du}{6} = (z+1)\,dz$$

$$\frac{1}{6}\int u^{-1/3}\,du = \frac{1}{6}\,\frac{u^{2/3}}{2/3} + C$$

$$= \frac{1}{4}\left(3z^2+6z+5\right)^{2/3} + C$$

<u>attempt #2</u> include power (not always recommended.

$$u = (3z^2+6z+5)^{1/3} \quad \text{— TRICK}$$

$$\Rightarrow u^3 = 3z^2+6z+5$$

$$3u^2\,du = (6z+6)\,dz$$

$$\frac{u^2}{2}\,du = (z+1)\,dz$$

$$\int \frac{1}{u}\,\frac{u^2}{2}\,du = \frac{1}{2}\int u\,du$$

$$= \frac{1}{2}\,\frac{u^2}{2} + C = \frac{1}{4}u^2 + C$$

$$= \frac{1}{4}\left(3z^2+6z+5\right)^{2/3} + C$$

both answers the same.

INTEGRATING TRIG FUNCTIONS

p253

From section 3.5, we have

$$d(\sin u) = \cos u \, du$$

and

$$d(\cos u) = -\sin u \, du$$

From these formulae, we get the basic integration rules

Thm

$$\int \cos u \, du = \sin u + C$$

$$\int \sin u \, du = -\cos u + C$$

Eg

$$\int (1 - \sin^2 3t) \cos 3t \, dt$$

$$= \int \cos 3t \, dt - \int \sin^2 3t \cos 3t \, dt$$

$$= I_1 - I_2$$

$$I_1 = \int \cos 3t \, dt \qquad \text{let } u = 3t$$

$$du = 3\,dt$$

$$= \frac{1}{3}\int \cos u \, du \qquad \frac{du}{3} = dt$$

$$= \frac{\sin u}{3} + C_1 = \frac{\sin 3t}{3} + C_1$$

$$I_2 = \int \sin^2 3t \cos 3t \, dt \qquad \text{let } u = \sin 3t$$

$$du = 3\cos 3t \, dt$$

$$\frac{du}{3} = \cos 3t \, dt$$

NOTE: NO trig function

$$= \frac{1}{3}\int u^2 \, du$$

$$= \frac{u^3}{9} + C_2 = \frac{\sin^3 3t}{9} + C_2$$

$$\therefore I_1 - I_2 = \frac{\sin 3t}{3} - \frac{\sin^3 3t}{9} + C$$

OR

$$\int (1 - \sin^2 3t) \cos 3t \, dt \qquad u = \sin 3t$$

$$du = 3\cos 3t \, dt$$

$$= \frac{1}{3}\int 1 - u^2 \, du \qquad \frac{du}{3} = \cos 3t \, dt$$

$$= \frac{u}{3} - \frac{u^3}{3 \cdot 3} + C = \frac{\sin 3t}{3} - \frac{\sin^3 3t}{9} + C$$

Sometimes creative insight is useful.

$$\int \frac{\cot x}{\sin x}\,dx \quad \rightarrow \text{change to } \sin x + \cos x.$$

$$= \int \frac{\cos x}{\sin^2 x}\,dx = \int \sin^{-2} x \cos x\,dx \qquad \begin{aligned} u &= \sin x \\ du &= \cos x\,dx \end{aligned}$$

$$= \int u^{-2}\,du = -u^{-1} + C$$

$$= -(\sin x)^{-1} + C = \underline{-\csc x} + C \,//$$

Sometimes a second substitution is necessary (or helpful)

EX 11 p255

$$\int 16x \sin^3(2x^2+1)\cos(2x^2+1)\,dx$$

$$\begin{aligned} u &= 2x^2+1 \\ du &= 4x\,dx \\ \frac{du}{4} &= x\,dx \end{aligned}$$

$$= \frac{1}{4}16\int \sin^3 u \cos u\,du$$

$$\begin{aligned} v &= \sin u \\ dv &= \cos u\,du \end{aligned}$$

$$= 4\int v^3\,dv = \cancel{4}\frac{v^4}{\cancel{4}} + C = v^4 + C$$

$$= \sin^4 u + C = \sin^4(2x^2+1) + C \,//$$

TRIG FUNCTION + ITS DERIVATIVE COMBINATION

In general, if we find a trig function and its derivative together forming an integrand, it is an easy task to integrate after a substitution.

EX 5 p 255 7a) $\int \sin^n x \cos x \, dx \quad (n \neq 1)$

$$= \int u^n du \qquad \begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}$$

$$= \frac{u^{n+1}}{n+1} + C = \frac{\sin^{n+1} x}{n+1} + C$$

Similarly

$$\int \cos^n x \sin x \, dx = -\frac{\cos^{n+1} x}{n+1} + C$$

(cf EX 6)

INGENUITY

Sometimes one needs insight to solve a problem.
At other times, one needs to remember an identity.

① EX 5

$$\int x^2 e^{x^3}\,dx$$

$\Rightarrow$ x^3 as exponent makes me nervous.

I know $\int e^u\,du = e^u + C$

but here, the exponent doesn't match the differential.

$\Rightarrow$ get rid of it!

① $\underline{u = x^3}$

② $\dfrac{du}{dx} = 3x^2$

③ $\underline{\dfrac{du}{3} = x^2\,dx}$

$$\Rightarrow \frac{1}{3}\int e^{u}\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

② Ex 6b $\int \sec x\,dx$

$\Rightarrow$ we know $\int \sec^2 x\,dx = \tan x + C$

but we don't know about $\int \sec x\,dx$

$\Rightarrow$ let's multiply by 1 written as $\dfrac{\sec x + \tan x}{\sec x + \tan x}$

$$\Rightarrow \int \sec x\,dx = \int \sec x \,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x}\,dx$$

Careful look suggests that the deriv of denom is the numer.

$\therefore$ let $u = \sec x + \tan x$

$\dfrac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = (\sec^2 x + \sec x \tan x)\,dx$

$$\Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

EX 9 Sometimes we need to remember trig formulas/identities that have nothing to do (directly) with calculus

EG:

$$\left.\begin{aligned} \sin\frac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} \\ \cos\frac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{2}} \end{aligned}\right\}$$ "Half-angle formulas

Also written as

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

Thus:

Ex 9a:

$$\int \sin^2 x\,dx = \frac{1}{2}\int 1-\cos 2x\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

OTHER TRIG IDENTITIES

p344 pg #79

Sometimes other identities must be used to obtain answers given in the book.

Skipped 2005

a) $\int 2\sin x\cos x\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$= 2\int u\,du = \frac{\cancel{2}u^2}{\cancel{2}} + C = \sin^2 x + C_1$ //

b) $\int 2\sin x\cos x\,dx$ $\quad u = \cos x$, $du = -\sin x\,dx$

$= -2\int u\,du = -\frac{\cancel{2}u^2}{\cancel{2}} + C = -\cos^2 x + C_2$ //

c) $\int 2\sin x\cos x\,dx$ $\quad$ use double angle formula $\sin 2x = 2\sin x\cos x$

$= \int \sin 2x\,dx$ $\quad u = 2x$, $du = 2\,dx$

$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos 2x + C_3$ //

Are all 3 answers correct? Why (not)?

__NOTE:__ The constants are all different

For a) b) remember $\sin^2 x + \cos^2 x = 1$

$$\therefore \quad \sin^2 x = 1 - \cos^2 x$$

$$\therefore \quad \sin^2 x + C_1 = -\cos^2 x + \underbrace{1 + C_1}$$
$$= -\cos^2 x + C_2$$

$$\therefore \quad \underline{C_2 = 1 + C_1}$$

For c) remember $\cos 2x = \cos^2 x - \sin^2 x$

$$\therefore \quad -\frac{1}{2}\cos 2x = -\frac{1}{2}\cos^2 x + \frac{1}{2}\sin^2 x$$
$$= -\frac{1}{2}\cos^2 x + \frac{1}{2}\left(1 - \cos^2 x\right)$$
$$= -\frac{1}{2}\cos^2 x - \frac{1}{2}\cos^2 x + \frac{1}{2}$$
$$= -\cos^2 x + \frac{1}{2}$$

$$\therefore \quad -\frac{1}{2}\cos 2x + C_3 = -\cos^2 x + \frac{1}{2} + C_3 = -\cos^2 x + C_2$$

$$\therefore \quad \underline{C_3 = C_2 - \frac{1}{2}}.$$