96-19-0

12(4,5) p 254
~~7(3.8) p 216~~

INDETERMINATE FORMS

Back to chpt 2 2.2 p 15

$$\lim_{x \to 1} \frac{x^2-1}{x-1}$$

was our favorite "problem" function and "problem" limit.
since direct substitution led to $\frac{0}{0}$.

__Def__ If f and g are continuous at $x=a$ and
$f(a)=g(a)=0$, then the limit of the quotient of functions

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be evaluated directly and is called an
__indeterminate form__.

In chapter 2, we learned algebraic methods to transform
indeterminate forms to usable forms for evaluation,

eg

$$\lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} x+1 = 2.$$

In this section we try another approach.

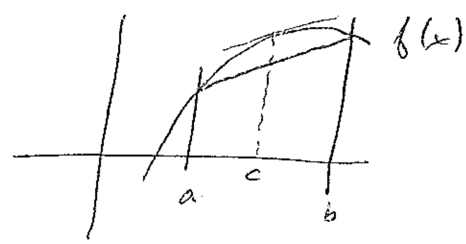
REVIEW: MEAN VALUE THM

(Th. 4) p 231

<u>Mean Value Thm</u>. Let $f(x)$ be cont's on $x \in [a, b]$ and differentiable on $x \in (a, b)$. Then $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, if a curve $f(x)$ is "nice" between $x = a$ and $x = b$, there is a point c between a and b s.t. the slope of the tangent line to f is the same as the slope of the straight line from a to b on the curve.



<u>Cauchy's MVT</u> (~~3-9 Thm 5~~) 12 Thm 7 p 259

Let $f(x)$ and $g(x)$ be cont's on $x \in [a, b]$ and diff'ble on $x \in (a, b)$. Let $g'(x) \neq 0$ for $x \in (a, b)$. Then $\exists\, c \in (a, b)$ st.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(proof in book – apply reg MVT twice)

~~(Stroock lectures)~~ (7B)

L'Hôpital's Rule (Abbott) Thm 6 p255

Both uses a instead of x_0 and interval is I

Let $f(x_0)=g(x_0)=0$. Let f, g be both diff'ble for $x\in(a,b)$ except possibly at $x_0\in(a,b)$.
Suppose $g'(x)\neq 0\ \forall x\neq x_0,\ x\in(a,b)$.

Then $$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

PROVIDED $\dfrac{f'(x)}{g'(x)}$ has a limit as $x\to x_0$. NOT Derivative

~~7 p A-17~~
8 B A-26
12 p 266

Partial proof (for case $x\to x_0^+$, also need proof for case $x\to x_0^-$)

Let $x>x_0$. By assumption $g'(x)\neq 0$. Apply Cauchy MVT to $[x_0,x]$. I.e. $\exists c\in(x_0,x)$ s.t.

$$\frac{f'(c)}{g'(c)}=\frac{f(x)-f(x_0)}{g(x)-g(x_0)}$$

Since $f(x_0)=g(x_0)=0$, we get

$$\frac{f'(c)}{g'(c)}=\frac{f(x)}{g(x)}$$

As $x\to x_0^+$, since $c\in(x_0,x)$, $c\to x_0^+$

$$\therefore \lim_{x\to x_0^+}\frac{f(x)}{g(x)}=\lim_{c\to x_0^+}\frac{f'(c)}{g'(c)}=\lim_{x\to x_0^+}\frac{f'(x)}{g'(x)}$$

Similarly for $x\to x_0^-$.
Both part together give the total proof.

USING L'HÔPITAL'S RULE

The main use is for indeterminate limits, e.g.

$0/0$ or ∞/∞ or $\infty \cdot 0$ or $\infty - \infty$

EG 1 $\lim_{x \to 0} \frac{\sin x}{x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$ //

7 219
% P 39
P 222
EX 2

EG 2 $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} \left(\frac{0}{0}\right)$

12 10
P 255

$= \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} \left(\frac{0}{0}\right)$ (again)

$= \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\frac{1}{8}$ //

12 E/2
P 256

EG 3 Don't go too far!

$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0$ //

(note $\neq \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$)

diff from
EX 4

EG 4 $\lim_{t \to \infty} \frac{t^2 + t}{2t^2 + 1} \left(\frac{\infty}{\infty}\right)$

$= \lim_{t \to \infty} \frac{2t + 1}{4t} \left(\frac{\infty}{\infty}\right)$

$= \lim_{t \to \infty} \frac{2}{4} = \frac{1}{2}$ //

EX 6
p 258

EG 5 $[\infty - \infty]$

$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) =$ (common denom) $\lim_{x \to 0} \left(\frac{x - \sin x}{x \sin x}\right) \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\sin x}{2\cos x - x \sin x} = 0$ //

This rule will be used in chpt 18 — series, ...