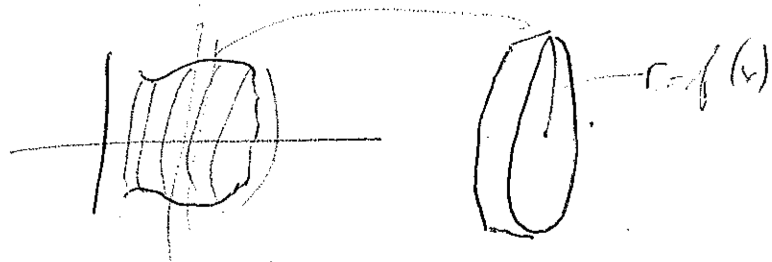
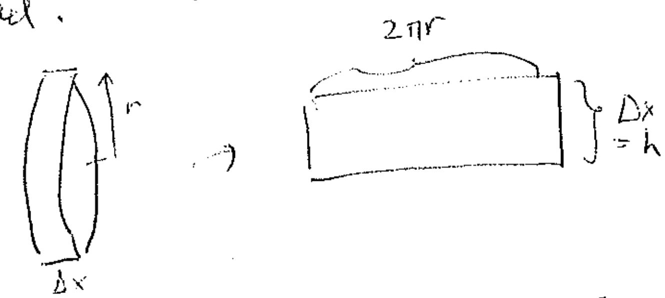
AREA OF SURFACE OF REVOLUTION ~~(8.2)~~

Take a curve $y = f(x)$ and revolve it around the x-axis. Instead of finding the volume of the solid generated, let's try to find the surface area.

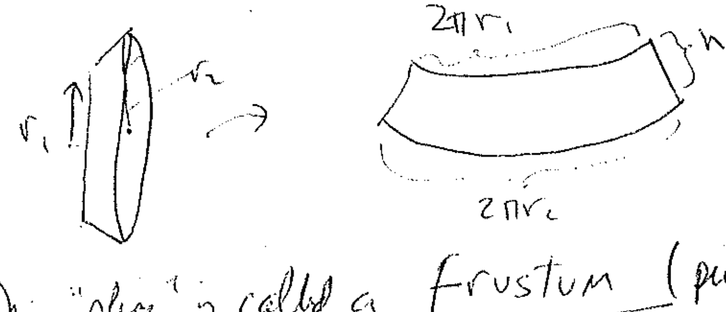


If the top edge were horizontal, we could slit this ring. It would be a "cylinder" and the surface area could easily be calculated.



$\Rightarrow A = 2\pi r h.$

But suppose we want a more accurate evaluation if the top were not horizontal, e.g. if the slice looks like a lampshade



This "slice" is called a frustum (piece of bit).

To get area we take the average of the two radii and get

$$A = \frac{2\pi r_1 + 2\pi r_2}{2} \cdot h = \cancel{\frac{2\pi(r_1+r_2)h}{2}} = \pi(r_1+r_2)h$$

In this case, however, $h \neq \Delta x$ since h is slanted.

$$\therefore h = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

Using the same approach as before.

$$A_{exact} = \lim_{n \to \infty} \sum \pi (r_i + r_{i+1}) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

[as n gets bigger $r_i = r_{i+1} = f(x_i) = r$

$$= \lim_{n \to \infty} 2\pi \sum f(x_i) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_b^b r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

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$y = \frac{x^3}{3} + \frac{1}{4x}$ $\quad 1 \le x \le 3 \quad$ rotate about $\underline{y=-1}$

$\Rightarrow \frac{dy}{dx} = \cancel{\frac{3x^2}{3}} + (-1)\frac{1}{4}x^{-2} = x^2 - \frac{1}{4x^2} = \frac{4x^4-1}{4x^2}$

$1+\left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{4x^4-1}{4x^2}\right)^2 = 1 + \frac{16x^8-8x^4+1}{16x^4} = \frac{16x^4+16x^8-8x^4+1}{16x^4}$

$= \frac{16x^8+8x^4+1}{16x^4} = \left(\frac{4x^4+1}{4x^2}\right)^2$

$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{4x^4+1}{4x^2}\right)^2} = \frac{4x^4+1}{4x^2} = x^2+\frac{1}{4}x^{-2}$

$A = 2\pi \int_1^3 r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= 2\pi \int_1^3 \left(\frac{x^3}{3}+\frac{1}{4}x^{-1}+1\right)\left(x^2+\frac{1}{4}x^{-2}\right)dx$

$= 2\pi \int_1^3 \frac{1}{3}x^5 + \frac{1}{12}x + \frac{1}{4}x + \frac{1}{16}x^{-3} + x^2 + \frac{1}{4}x^{-2}\,dx$

$= 2\pi \int_1^3 \frac{1}{3}x^5 + x^2 + \frac{1}{3}x + \frac{1}{4}x^{-2} + \frac{1}{16}x^{-3}\,dx$

$= 2\pi \left[\frac{1}{3}\frac{x^6}{6} + \frac{x^3}{3} + \frac{1}{3}\frac{x^2}{2} + \frac{1}{4}\frac{x^{-1}}{(-1)} + \frac{1}{16}\frac{x^{-2}}{(-2)}\right]_1^3$

$= 2\pi \left[\frac{x^6}{18} + \frac{x^3}{3} + \frac{x^2}{6} - \frac{x^{-1}}{4} - \frac{x^{-2}}{32}\right]_1^3$

$= 2\pi \left[\frac{3^6}{18} + \frac{3^3}{3} + \frac{3^2}{6} - \frac{3^{-1}}{4} - \frac{3^{-2}}{32} - \left(\frac{1}{18}+\frac{1}{3}+\frac{1}{6}-\frac{1}{4}-\frac{1}{32}\right)\right]$

$= 2\pi \left[\frac{81}{2} + 9 + \frac{3}{2} - \frac{1}{12} - \frac{1}{9\cdot 32} - \frac{1}{18} - \frac{1}{3} - \frac{1}{6} + \frac{1}{4} + \frac{1}{32}\right]$

$= 2\pi \left[42 + 9 - \frac{1}{12} + \frac{1}{36} - \frac{5}{9} + \frac{1}{4}\right]$

$= 2\pi \left[51 + \frac{-3-20+1+9}{36}\right] = 2\pi\left[51 - \frac{13}{36}\right] = \frac{1823}{18}\pi$

EXAMPLE

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Ex 1
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[illegible]

Given $y = 2\sqrt{x}$ with $1 \le x \le 2$, find area of surface of revolution.

$$y = 2\sqrt{x} = 2x^{1/2} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{2} x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{\sqrt{x}}\right)^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)} = \sqrt{\frac{x+1}{x}}$$

$$A = 2\pi \int_1^2 2\sqrt{x} \sqrt{\frac{x+1}{x}}\, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1}\, dx = 4\pi \int_1^2 (x+1)^{1/2} dx$$

$$= 4\pi \frac{2(x+1)^{3/2}}{3} \Bigg]_1^2$$

$$= \frac{8\pi}{3}\left[(3)^{3/2} - (2)^{3/2}\right]$$

$$= \frac{8\pi}{3}\left[3\sqrt{3} - 2\sqrt{2}\right]$$