(89-24)

IMPROPER INTEGRALS? ~~(7.8)~~ (8,7)

Look at the integral

$$\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\, dx$$

By an appropriate ~~substitution~~ trick, we can solve it.

But notice that $f(x) = \sqrt{\frac{1+x}{1-x}}$ is not defined at one of the limit of integration, $x = 1$ (i.e. $f(x) \to \infty$ as $x \to 1$).

This is one type of integral which is called <u>improper</u>.

<u>Def</u> In general,

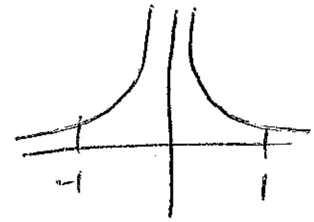
(A) if $\int_a^b f(x)\,dx$ is s.t. $f(x) \to \infty$ for some $x \in [a, b]$ Type II p 499

or (B) if we have $\int_a^\infty f(x)\,dx$ or $\int_\infty^b f(x)\,dx$ Type I p 496

we say that $\int_a^b f(x)\,dx$ is an <u>improper integral</u>.

BEWARE OF IMPROPER INTEGRALS

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$



positive area between curve and x-axis.

Try approach we've used thus far.

$$\int_{-1}^{1} x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right]_{-1}^{1}$$

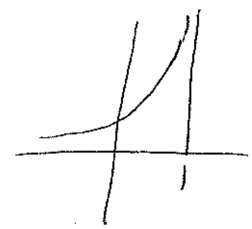
$$= \left.-\frac{1}{x}\right]_{-1}^{1} = -\frac{1}{(1)} - \frac{-1}{(-1)} = -1-1 = \underline{\underline{-2}}$$ ← inconsistent value!

the old approach (not taking the infinite height into account) does $\underline{\text{NOT}}$ work!

B EVALUATING IMPROPER INTEGRALS: $f(x) \to \infty$

<u>Def</u> Suppose $f(x) \to \infty$ as $x \to 1$

Look at the integral $\int_{-1}^{b} f(x)\,dx$



If this integral has a <u>finite</u> limit as $b \to 1$ from inside the interval, then we say that the integral <u>converges</u>, and we define

$$\int_{-1}^{1} f(x)\,dx = \lim_{b \to 1^-} \int_{-1}^{b} f(x)\,dx.$$

If this integral has <u>no</u> finite limit, then we say the integral <u>diverges</u>.

EX 1) Does $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$ converge or diverge? Problem is at upper limit 1.

$$\int_0^b \sqrt{\frac{1+x}{1-x}}\,dx = \Big[\arcsin x - \sqrt{1-x^2}\,\Big]_0^b$$

$$= \left(\arcsin b - \sqrt{1-b^2}\right) - \left(\arcsin \overset{0}{0} - \overset{1}{\sqrt{1-0}}\right)$$

$$= \arcsin b - \sqrt{1-b^2} + 1$$

Now as $b \to 1$ $\arcsin b \to \frac{\pi}{2}$ and $\sqrt{1-b^2} \to 0$

$$\therefore \lim_{b \to 1^-} \int_0^b \sqrt{\frac{1+x}{1-x}}\,dx = \frac{\pi}{2} - 0 + 1 = \underline{\frac{\pi}{2} + 1} < \infty$$

$\therefore$ <u>converges</u>

$$\int_0^b \sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1+x}{1+x}}\,dx = \int_0^b \frac{1+x}{\sqrt{1-x^2}}\,dx = \int_0^b \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$$

EX 2

TW p 493

54

What about $\int_0^1 \frac{dx}{x}$? Here the problem is at the lower limit 0.

as $x \to 0$, $\frac{1}{x} \to \infty$.

$$\int_b^1 \frac{dx}{x} = \log x \Big]_b^1 = \cancel{\log 1}^{0} - \log b = \log \frac{1}{b}$$

as $b \to 0$, $\frac{1}{b} \to \infty$ and therefore $\log \frac{1}{b} \to \infty$

$\therefore \int_0^1 \frac{dx}{x}$ diverges.

EX ~~8~~ 5

~~TW p 493~~

12 500

What about $\int_0^3 \frac{dx}{(x-1)^{2/3}}$? Here the problem is inside the interval of integration, at $x=1$.

as $x \to 1$, $\frac{1}{x-1} \to \infty$ $\therefore$ $\frac{1}{(x-1)^{2/3}} \to \infty$.

Break it up into 2 integrals

eg. $\int_0^b \frac{dx}{(x-1)^{2/3}}$ and $\int_c^3 \frac{dx}{(x-1)^{2/3}}$

Check each integral.

<u>Note:</u> $\int \frac{dx}{(x-1)^{2/3}} = 3(x-1)^{1/3} + C$

$$\lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim \left[3(x-1)^{1/3}\right]_0^b = \lim \left[3(\cancel{b-1})^{1/3} - 3(-1)^{1/3}\right]$$

$= 0 - (-3) = +3$ finite !!

$$\lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim \left[3(x-1)^{1/3}\right]_c^3 = \lim \left[3(3-1)^{1/3} - 3(\cancel{c-1})^{1/3}\right]$$

$= 3\sqrt[3]{2}$ finite also !!

$$\therefore \int_0^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} + \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$= 3 + 3\sqrt[3]{2}$ // Converges

C. EVALUATING IMPROPER INTEGRALS: One limit $= \infty$.

Look at the 2nd type of improper integral — those in which one of the limits of integration is ∞.

~~Ex 6 p524~~ ~~Ex 4 p494~~

Look at the following integrals & determine convergence/divergence

Ⓐ $\displaystyle\int_1^\infty \frac{dx}{x}$ Ⓑ $\displaystyle\int_1^\infty \frac{dx}{x^2}$

cf. EX 3 p498

Ⓐ $$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \log x \Big]_1^b$$

$$= \lim_{b\to\infty} [\log b - \log 1^{\,0}]$$

$$= \lim_{b\to\infty} \log b \to \infty \quad \underline{\text{diverges}} \ !$$

Ⓑ $$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \frac{-1}{x}\Big]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right] \to 1 \quad \underline{\text{converges}} \ !$$

– REMEMBER THESE 2 EXAMPLES!

– THEY WILL BE COMMONLY USED IN COMPARISONS

In general, for $I = \displaystyle\int_1^\infty \frac{dx}{x^p}$

if $p > 1$, I converges

if $p \le 1$, I diverges

What happens if you can't integrate it?

E6. Does $\int_1^\infty e^{-x^2}\,dx$ converge or diverge?

~~EX 4 p495~~

~~Let $I(b) = \int_1^b e^{-x^2}\,dx$~~

12 EX 6 p501

Let's look at a larger function & see if it converges

Now $\forall x > 1, \quad x < x^2$

$\therefore \quad e^x < e^{x^2}$

$\therefore \quad \frac{1}{e^x} > \frac{1}{e^{x^2}}$

i.e. $e^{-x} > e^{-x^2}$

$$\therefore \quad \int_1^b e^{-x}\,dx > \int_1^b e^{-x^2}\,dx$$

Now $\int_1^b e^{-x}\,dx = \left[-e^{-x}\right]_1^b = -e^{-b} + e^{-1}$

as $b \to \infty$ we get $-\frac{1}{e^b} \to 0$

$\therefore$ as $b \to \infty$ $\int_1^b e^{-x}\,dx \to \frac{1}{e} = .367879441$

$$\therefore \quad \int_1^\infty e^{-x^2}\,dx \leq \lim_{b\to\infty} \int_1^b e^{-x}\,dx = .36787$$

$\therefore$ converges.

This leads to formulating the following Thm

Thm 2 (~~Domination Test~~) (Direct Comparison Test)

~~7 §5.4 (p 496)~~
~~8 §5.5 (p 5..)~~
12 p 502

If $0 \le f(x) \le g(x) \quad \forall x \geq a$

then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ ~~converges~~ diverge.

MORE ON IMPROPER INTEGRALS

Thm 3 (Limit Comparison Test)

~~? p 497~~ ~~8 520~~ 12 p502

Suppose both $f(x)$ and $g(x)$ are positive functions and $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L$ where $0 < L < \infty$

Then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

~~Ex 9~~ ~~Ex 7~~ ~~p 524~~ ~~p497~~ Ex 3 p 503

Compare $\int_1^\infty \dfrac{dx}{x^2}$ and $\int_1^\infty \dfrac{dx}{1+x^2}$.

Let $f(x) = \dfrac{1}{x^2}$, $g(x) = \dfrac{1}{1+x^2}$

Then $\lim\limits_{x\to\infty} \dfrac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim \dfrac{1+x^2}{x^2} = \lim\left(\dfrac{1}{x^2}+1\right) = 0+1 = 1$

This is a positive finite limit.

Since we know that $\int_1^\infty \dfrac{dx}{x^2}$ converges,

$\therefore \int_1^\infty \dfrac{dx}{1+x^2}$ also converges.

NOTE: The 2 integrals may converge to different values!

OTHER WAYS TO DIVERGE

7 EX 10 / B 499 — 8 12 / 528

$$\int_0^b \cos x\, dx = \sin x \Big]_0^b = \sin b - \sin 0 = \sin b.$$

As $b \to \infty$, $\sin b$ goes from $+1$ to -1 and back.

But it takes on NO one value.

It constantly oscillates, but it does not converge.

$\therefore$ This limit does not exist and thus

$$\int_0^\infty \cos x\, dx \text{ does not exist.}$$

This can be called divergence by oscillation.

NOT in 12R