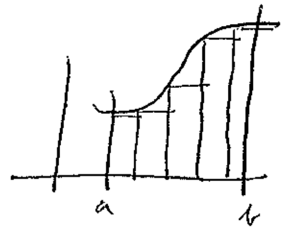
A. APPROXIMATING INTEGRALS (8.6 ~~(4.4)~~) p 486, p 290

We think of integrals as areas under curves, i.e.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \text{where } \Delta x = \frac{b-a}{n}$$

This uses "rectangles" i.e.



We may get better approximation using other alternatives.

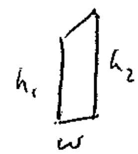
2 major choices – ① trapezoids, ② parabolas

⇒ ① trapezoidal rule

② Simpson's rule.

B. TRAPEZOIDAL RULE

What is area of trapezoid?

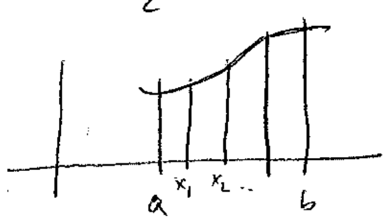


$$A_T = \left(\frac{h_1+h_2}{2}\right) w$$

or $\quad \dfrac{y_{i-1}+y_i}{2}\,\Delta x$

Given curve

Let $y_i = f(x_i)$



$$\text{Total area (trapez)} = \frac{y_0+y_1}{2}\Delta x + \frac{y_1+y_2}{2}\Delta x + \dots + \frac{y_{n-1}+y_n}{2}\Delta x$$

$$= \left(\frac{y_0}{2} + \frac{y_1}{2} + \frac{y_1}{2} + \frac{y_2}{2} + \dots + \frac{y_{n-1}}{2} + \frac{y_n}{2}\right)\Delta x$$

$$= \left(\frac{y_0}{2} + y_1 + y_2 + \dots + y_{n-1} + \frac{y_n}{2}\right)\Delta x$$

$$= \left(\frac{y_0}{2} + \frac{y_n}{2}\right)\Delta x + \sum_{i=1}^{n-1} y_i\,\Delta x = \frac{\Delta x}{2}\left(y_0 + y_n + 2\sum_{i=1}^{n-1} y_i\right)$$

or if $\Delta x = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(y_0 + y_n + 2\sum_{i=1}^{n-1} y_i\right) = A_T$$

EX 1

pg ~~218~~ ~~299~~

~~#~~ ~~#3~~

$$\int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right]_0^2 = \frac{16}{4} = \underline{\underline{4}}$$

Let $n = 4$ $\therefore$ $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

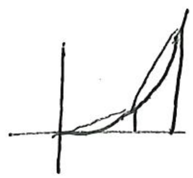
$y_0 = f(0) = 0$
$y_1 = f(\frac{1}{2}) = \frac{1}{8}$
$y_2 = f(1) = 1$
$y_3 = f(\frac{3}{2}) = \frac{27}{8}$
$y_4 = f(2) = 8$

$$A_T = \left(\frac{1}{2}\right)\cdot\frac{1}{2}\left(0 + 8 + 2\left(\frac{1}{8} + 1 + \frac{27}{8}\right)\right)$$

$$= \frac{1}{4}\left(8 + \frac{2\cdot 36}{8}\right) = \frac{1}{4}(8+9)$$

$$= \frac{17}{4} = \underline{\underline{4\tfrac{1}{4}}}$$

Note: approximation is greater!



NOTE: Computers ~~cannot~~ couldn't do analytic integration! Formerly — Now Derive, Mathematica, Maple

also there exist ~~with~~ functions which are almost impossible to integrate analytically

$\therefore$ we must use approximations!

NOTE: It's possible to do error analysis on this method. Better approximation with higher n.

C. SIMPSON'S RULE – PARABOLAS (p ~~295~~ 488)

Look at general parabola: $y = Ax^2 + Bx + C$

Find exact area under parabola from $x = -h$ to $x = h$.

$$A_p = \int_{-h}^{h} Ax^2 + Bx + C \, dx$$

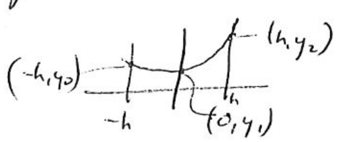
$$= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big]_{-h}^{h}$$

$$= \left(\frac{Ah^3}{3} + \frac{Bh^2}{2} + Ch\right) - \left(-\frac{Ah^3}{3} + \frac{Bh^2}{2} - Ch\right)$$

$$= \frac{2Ah^3}{3} + 2Ch \, .$$

Assume parabola goes through pts $(-h, y_0)$, $(0, y_1)$, (h, y_2)

i.e.



then
$$y_0 = Ah^2 - Bh + C$$
$$y_1 = C$$
$$y_2 = Ah^2 + Bh + C$$

Then
$$y_0 - y_1 = Ah^2 - Bh$$
$$y_2 - y_1 = Ah^2 + Bh$$

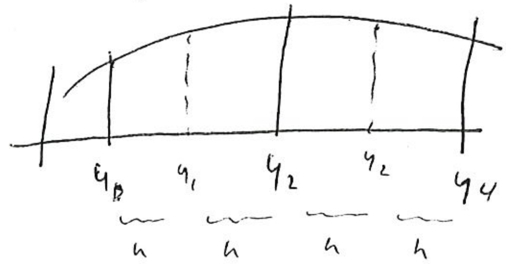
adding $\therefore$ $y_0 + y_2 - 2y_1 = 2Ah^2$

From above $A_p = \dfrac{2Ah^3}{3} + 2Ch = \left(2Ah^2 + 6C\right)\dfrac{h}{3}$

$$= \left(\left[y_0 + y_2 - 2y_1\right] + 6y_1\right)\frac{h}{3}$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Let's cut curve in half.



approximate by 2 parabolas, one through y_0, y_1, y_2
the other through y_2, y_3, y_4

$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

We can expand this to a general form.

$$A_S = A_P = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

where n is an EVEN number. $\left(\Delta x = h = \frac{b-a}{n}\right)$

EX ~~29~~ 3
~~B 28~~ #~~4~~ again.

$$\int_0^2 x^3\,dx = \underline{\underline{4}}$$

$y = f(x) = x^3$
$y_0 = f(0) = 0$
$y_1 = f(\tfrac{1}{2}) = \tfrac{1}{8}$
$y_2 = f(1) = 1$
$y_3 = f(\tfrac{3}{2}) = \tfrac{27}{8}$
$y_4 = f(2) = 8$

Let $n = 4$

$$h = \frac{2-0}{4} = \frac{1}{2}$$

$$A_S = \frac{1}{2}\cdot\frac{1}{3}\left[0 + 4\cdot\frac{1}{8} + 2\cdot 1 + 4\cdot\frac{27}{8} + 8\right]$$

$$= \frac{1}{6}\left[\frac{1}{2} + 2 + \frac{27}{2} + 8\right]$$

$$= \frac{1}{6}\left[\frac{28}{2} + 10\right] = \frac{1}{6}\left[14 + 10\right] = \frac{1}{6}\left[24\right] = \underline{\underline{4}}$$

M is upper bound for $|f''|$ (Trap) and $|f^{(IV)}|$ (Simps)

NOTES Can also do error analysis on this method.

T error is related to f'' (pg 2V(A)) ~~19~~ ~~293~~ 8th 7th Thm 1 p 490 Box

S error is related to $f^{(IV)}$ (pg 2V(S)) ~~303~~ ~~297~~

$\therefore$ if f^{IV} is 0, no error!

$\therefore$ since $\frac{d^4}{dx^4} x^3 = 0$, we get 0 error using Simpson.

~~END CHPT 4~~

HW ~~pg 2V(A) #1-4 1st~~ ~~with n=4, using T & S & Exact (i.e. a, b, c)~~

~~END cpt 4~~

EX computing max possible error.

Look at T error.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}$$

where M is $\max|f''|$ on $[a,b]$

analysis of problem $\int_0^2 x^3\,dx$

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$$

$$\Rightarrow \text{on } [0,2],\ M = \max|f''| = f''(2) = 6\cdot 2 = \underline{12}$$

$$\therefore |E_T| \le \frac{12(2-0)^3}{12(4)^2} = \frac{8}{16} = \frac{1}{2}$$

The actual error was only $\underline{\underline{\frac{1}{4}}}$.

EX 4 p492

Estimate the min # of subintervals needed to approx $\int_0^2 5x^4\,dx$ using Simpson's Rule with error mag $< 10^{-4}$.

$$f(x) = 5x^4 \Rightarrow f'(x) = 20x^3 \Rightarrow f'' = 60x^2$$

$$\Rightarrow f''' = 120x \Rightarrow \underline{f^{iv} = 120 = M}$$

$$\therefore \; |E_s| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2-0)^5}{180n^4} = \frac{120 \cdot 2^5}{180n^4} = \frac{2 \cdot 2^5}{3n^4} = \frac{2^6}{3n^4}$$

we want this to be $< 10^{-4}$

$$\therefore \; \frac{2^6}{3n^4} < \frac{1}{10^4}$$

$$\Rightarrow \frac{3n^4}{2^6} > 10^4$$

$$\Rightarrow n^4 > \frac{64 \cdot 10^4}{3}$$

$$\Rightarrow n > 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5$$

$\therefore$ $n = 22$ (even #) at min to get the desired error tolerance