89-23

PARTIAL FRACTIONS ~~8.12~~ ~~7.5~~ ~~(7.7)~~ [12] 8.4 p 471

Look at $\frac{17}{35}$. In theory it can be divided into the sum of $\frac{A}{5}$ and $\frac{B}{7}$.

$$\frac{17}{35} = \frac{A}{5} + \frac{B}{7} = \frac{7A+5B}{35}$$

Let $A=1$, $B=2$ let $A=6$, $B=-5$

This is not uniquely determined in this case, But it can be unique in other cases.

~~7.R~~ of EX1 ~~p 482~~ 8.R ~~508~~ 12 p 471

Look at $\frac{5x-3}{x^2-2x-3}$. We can factor the bottom into $(x+1)(x-3)$.

$\therefore = \frac{5x-3}{(x+1)(x-3)}$. We would like to break this down into a sum of 2 fractions: $\frac{A}{x+1} + \frac{B}{x-3}$

$$\frac{A}{x+1} + \frac{B}{x-3} = \frac{5x-3}{(x+1)(x-3)}$$

Cross multiply

$$A(x-3) + B(x+1) = 5x-3$$
$$Ax - 3A + Bx + B = 5x-3$$
$$(A+B)x + (B-3A) = 5x-3$$

This is true iff like powers of x have identical coefficients.

i.e. $A+B=5$ — coeff of x

$-3A+B=-3$ — constant term.

Now solve the 2x2 linear system.

① − ② $\Rightarrow$ $4A = 8 \Rightarrow A = 2$

$\therefore B = 3$

$$\therefore \frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

$$\therefore \int \frac{5x-3}{x^2-2x-3}\,dx = 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3}$$

ADDITIONAL CASES (See BOY pp 472-73)

~~508~~ 12 473

#1 (cf EX 2 p ~~482~~) If one of the linear factors in the denominator is raised to a power i.e. $(x-r)^m$

then you must have <u>m</u> partial fractions on the other side whose denominators range from $(x-r)$ to $(x-r)^m$.

eg.

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

~~509~~ 474

#2 (cf EX 4 ~~p 483~~) If you have a non-factorable quadratic factor

eg. x^2+px+q

then this may be the sum of 2 partial fractions, one of which has an <u>x</u> in the numerator,

i.e.

$$\frac{A}{x^2+px+q} + \frac{Bx}{x^2+px+q}$$

NOTE: In practice, it is easier to use

$$\frac{A^*}{x^2+px+q} + \frac{B^*(2x+p)}{x^2+px+q}$$

deriv of denom

#3 EX5 p 475 If a quadratic factor is raised to a power, combine procedures in cases #1 and #2.

eg. $(x^2+px+q)^m$ corresponds to

$$\frac{A_1+B_1x}{x^2+px+q} + \frac{A_2+B_2x}{(x^2+px+q)^2} + \cdots$$

(cf EX 3 p 474)

cf. ~~EX6 p 484~~ #4 If the rational fraction $\frac{f(x)}{g(x)}$ is such that $\deg f(x) \geq \deg g(x)$, then <u>divide first</u> and then obtain $h(x) + \frac{f_2(x)}{g(x)}$ where $\deg f_2(x) < \deg g(x)$.

The factors of $g(x)$ <u>should</u> be known. In <u>theory</u>, any polynomial $g(x)$ with real coefficients can be expressed as a product of real linear + quadratic factors. In <u>practice</u>, it may be difficult to factor.

MORE EXAMPLES

NOTE: If you have a quadratic factor in the denominator, it is often easier to use its derivative as a factor in the numerator, rather than a plain x.

EX ~~p 489 #26~~ ~~512 #38~~ 12 p479 #21

$$\int_0^1 \frac{1\,dx}{(x+1)(x^2+1)}$$

deriv of bottom

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{B(2x)}{x^2+1} + \frac{C}{x^2+1}$$

$$1 = A(x^2+1) + B(2x)(x+1) + C(x+1)$$
$$= Ax^2 + A + 2Bx^2 + 2Bx + Cx + C$$

x^2: $0 = A + 2B$

x: $0 = 2B + C$

c: $1 = A + C$

$\Rightarrow 1 = A - 2B$, $1 = 2A \Rightarrow A = \frac{1}{2}$; $C = \frac{1}{2}$; $B = -\frac{1}{4}$

$$\therefore \int_0^1 \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \left(-\frac{1}{4}\right)\int_0^1 \frac{2x\,dx}{x^2+1} + \frac{1}{2}\int_0^1 \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\arctan x \Big]_0^1$$

$$= \frac{1}{4}\log(x+1)^2 - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\arctan x \Big]_0^1$$

$$= \frac{1}{4}\log\frac{(x+1)^2}{x^2+1} + \frac{1}{2}\arctan x \Big]_0^1$$

$$= \frac{1}{4}\log\frac{(1+1)^2}{1+1} + \frac{1}{2}\arctan 1 - \frac{1}{4}\log\frac{1}{1} - \frac{1}{2}\arctan 0$$

$$= \frac{1}{4}\log 2 + \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\cdot 0 - \frac{1}{2}\cdot 0$$

$$= \frac{\log 2}{4} + \frac{\pi}{8}$$

A. HEAVISIDE METHOD

quick method to use, but only for __linear__ factors

22-0
26-1
N25-4

~~16~~ 6

EX ~~#~~. ~~P3 360 486~~ 476

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

mult by $x-1$

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{(x-2)} + \frac{C(x-1)}{(x-3)}$$

let $x=1$ $\quad \frac{1+1}{(-1)(-2)} = A = \frac{2}{+2} = 1$

mult by $x-2$

$$\frac{x^2+1}{(x-1)(x-3)} = \frac{A(x-2)}{x-1} + B + \frac{C(x-2)}{x-3}$$

let $x=2$

$$\frac{5}{(1)(-1)} = B = -5$$

mult by $x-3$

$$\frac{x^2+1}{(x-1)(x-2)} = \frac{A(x-3)}{x-1} + \frac{B(x-3)}{x-2} + C$$

let $x=3$

$$\frac{10}{2\cdot 1} = C = 5$$

This is sometimes called the "Cover-ups" method, because, in practice, people just "cover-ups" one of the factors in the denominator while evaluating the other factors.

e.g.

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

Cover up $x-1$ while evaluating the rest of the expression at $x=1$.

$$\frac{1+1}{(1-2)(1-3)} = \frac{2}{-1\cdot -2} = \frac{2}{2} = \underline{1 = A}$$

etc.

MAYBE SKIP?

do this & skip part of next page?

EXAMPLES

~~EX 5 p 485~~ (GR) $\displaystyle\int \frac{x^5 - x^4 - 3x + 5}{x^4 - 2x^3 + 2x^2 - 2x + 1}\,dx$

~~cf. EX 6 p 484~~

cf EX 3 p 474

divide first.

$$\begin{array}{r|l} & x + 1 \\ \hline x^4-2x^3+2x^2-2x+1 & x^5 - x^4 + 0 + 0 - 3x + 5 \\ & x^5 - 2x^4 + 2x^3 - 2x^2 + x \\ \hline & x^4 - 2x^3 + 2x^2 - 4x + 5 \\ & x^4 - 2x^3 + 2x^2 - 2x + 1 \\ \hline & -2x + 4 \end{array}$$

$$\therefore = x + 1 + \frac{-2x+4}{x^4 - 2x^3 + 2x^2 - 2x + 1}$$

Can we factor the denom? Try dividing by $x-1$

$$\begin{array}{r|l} & x^3 - x^2 + x - 1 \\ \hline x-1 & x^4 - 2x^3 + 2x^2 - 2x + 1 \\ & x^4 - x^3 \\ \hline & -x^3 + 2x^2 - 2x + 1 \\ & -x^3 + x^2 \\ \hline & x^2 - 2x + 1 \end{array}$$

$x^3 - x^2 + x - 1 = (x-1)(x^2) + x - 1 = (x-1)(x^2+1)$

$x^2 - 2x + 1 = (x-1)^2$

GR cf EX4 p 494-5

$$\therefore \ x^4 - 2x^3 + 2x^2 - 2x + 1 = (x-1)^2(x^2+1)$$

skip?

$$\therefore \ \frac{-2x+4}{x^4 - 2x^3 + 2x^2 - 2x + 1} = \frac{Ax}{x^2+1} + \frac{B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\begin{aligned} -2x + 4 &= Ax(x-1)^2 + B(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) \\ &= Ax(x^2 - 2x + 1) + B(x^2 - 2x + 1) + C(x^3 - x^2 + x - 1) + D(x^2+1) \\ &= Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 - Cx^2 + Cx - C + Dx^2 + D \\ &= x^3(A + C) + x^2(-2A + B - C + D) + x(A - 2B + C) + (B - C + D). \end{aligned}$$

$0 = A + C$

$0 = -2A + B - C + D$

$-2 = A - 2B + C$

$4 = B - C + D$

$4 = 2A \Rightarrow \underline{A = 2}$

$\underline{C = -2}$

$-2 = 2 - 2B - 2$

$\underline{B = 1}$

$4 = 1 + 2 + D$

$\underline{1 = D}$

$$\therefore \int \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\,dx = \underset{I_1}{2\int \frac{x\,dx}{x^2+1}} + \underset{I_2}{\int \frac{dx}{x^2+1}} + \underset{I_3}{(-2)\int \frac{dx}{x-1}} + \underset{I_4}{\int \frac{dx}{(x-1)^2}}$$

$$I_1: \quad 2\int \frac{x\,dx}{x^2+1} \quad \begin{matrix} u = x^2+1 \\ du = 2x\,dx \end{matrix} \quad = \int \frac{du}{u} = \log|u| + C_1 = \log|x^2+1| + C_1$$

$$I_2: \quad \int \frac{dx}{x^2+1} = \arctan x + C_2$$

$$I_3: \quad -2\int \frac{dx}{x-1} \quad \begin{matrix} u = x-1 \\ du = dx \end{matrix} \quad = -2\int \frac{du}{u} = -2\log|u| + C_3 = -2\log|x-1| + C_3$$

$$I_4: \quad \int \frac{dx}{(x-1)^2} \quad \begin{matrix} u = x-1 \\ du = dx \end{matrix} \quad = \int \frac{du}{u^2} = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C_4 = \frac{-1}{x-1} + C_4$$

$$\therefore I_1 + I_2 + I_3 + I_4 = \log|x^2+1| + \arctan x - 2\log|x-1| - \frac{1}{x-1} + C$$

$$= \log \frac{x^2+1}{(x-1)^2} + \arctan x - \frac{1}{x-1} + C_5$$

$$\therefore \int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx = \int x+1\,dx + \int \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\,dx$$

$$= \frac{x^2}{2} + x + \log \frac{x^2+1}{(x-1)^2} + \arctan x + \frac{1}{1-x} + C \quad //$$

WHY ALL TERMS.

$$\frac{A}{x^2+1} + \frac{Bx}{x^2+1} + \frac{C}{(x^2+1)^2} + \frac{Dx}{(x^2+1)^2} = \frac{x^3}{(x^2+1)^2}$$

$$\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} = \frac{x^2+1}{(x^2+1)^2} + \frac{x}{(x^2+1)^2} = \frac{x^2+x+1}{(x^2+1)^2}$$

$$\frac{x}{x^2+1} + \frac{1}{(x^2+1)^2} = \frac{x(x^2+1)}{(x^2+1)^2} + \frac{1}{(x^2+1)^2} = \frac{x^3+x+1}{(x^2+1)^2}$$

$$\frac{1}{x^2+1} + \frac{x}{x^2+1} + \frac{1}{(x^2+1)^2} + \frac{x}{(x^2+1)^2} = \frac{(x^2+1)+(x^3+x)+1+x}{(x^2+1)^2}$$

$$= \frac{x^3+x^2+2x+2}{(x^2+1)^2}$$

$$\frac{1}{x^2+1} + \frac{x}{x^2+1} - \frac{1}{(x^2+1)^2} - \frac{x}{(x^2+1)^2} = \frac{(x^2\cancel{+1})+(x^3\cancel{+x})\cancel{-1}\cancel{-x}}{(x^2+1)^2}$$

$$= \frac{x^3+x^2}{(x^2+1)^2}$$

~~P5 489 #22~~

~~NST 4 8.8~~ p 479 #13

$$\int_4^8 \frac{y\,dy}{y^2-2y-3}$$

$$y^2-2y-3 = (y-3)(y+1)$$

$$\frac{y}{y^2-2y+3} = \frac{A}{y-3} + \frac{B}{y+1}$$

$$y = Ay + A + By - 3B$$

x: $1 = A+B$

consts: $0 = A-3B$

$$1 = 4B \Rightarrow B = \frac{1}{4}$$

$$\Rightarrow A = 3/4$$

$$\therefore \int_4^8 \frac{y\,dx}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dx}{y-3} + \frac{1}{4}\int_4^8 \frac{dx}{y+1}$$

$$= \frac{3}{4}\log|y-3| + \frac{1}{4}\log|y+1| \Big]_4^8$$

$$= \frac{3}{4}\log|5| + \frac{1}{4}\log 9 - \cancel{\frac{3}{4}\log 1}^{\,0} - \frac{1}{4}\log 5$$

$$= \frac{1}{2}\log 5 + \frac{1}{2}\cdot\frac{1}{2}\log 3^2$$

$$= \frac{1}{2}\log 5 + \frac{1}{2}\log 3$$

$$= \frac{1}{2}\log 15 = \log\sqrt{15}$$

EX

$$\int \sec\theta \, d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta}{\cos^2\theta}\, d\theta = \int \frac{\cos\theta \, d\theta}{1-\sin^2\theta}$$

Let $x = \sin\theta$
$dx = \cos\theta \, d\theta$

$$= \int \frac{dx}{1-x^2} \quad (\cancel{\text{cf } p.\,450 \ \#11}) \ ?$$

CB

$$1-x^2 = (1-x)(1+x)$$

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$1 = A(1+x) + B(1-x) = A + Ax + B - Bx$$
$$= (A-B)x + (A+B)$$

$\therefore \ 0 = A - B$ coef of x

$1 = A + B$ const.

add $1 = 2A \Rightarrow A = \frac{1}{2}$

$B = \frac{1}{2}$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x}$$

$u = 1-x$ $\quad v = 1+x$
$du = -dx$ $\quad dv = dx$

$$= -\frac{1}{2}\int \frac{du}{u} + \frac{1}{2}\int \frac{dv}{v}$$

$$= -\frac{1}{2}\log|u| + \frac{1}{2}\log|v| + C$$

$$= -\frac{1}{2}\log|1-x| + \frac{1}{2}\log|1+x| + C$$

$$= \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

?

CB

$$= \frac{1}{2}\log\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C$$

$$= \log\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + C$$

mult by $\frac{1+\sin\theta}{1+\sin\theta}$

$$= \log\sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} + C$$

$$= \log\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}} + C$$

$$= \log\left(\frac{1+\sin\theta}{\cos\theta}\right) + C$$

$$= \log|\sec\theta + \tan\theta| + C \ //$$

[483 compl. sq. # 1, 9, 11]

HW

1, 3, 5, 9

490 # 23, 25, 29

506 # 29