PRELIM

$$\int \frac{dx}{\sqrt{1-x^2}} \longrightarrow \text{arc}\sin x + C$$

$$\int \frac{x\,dx}{1-x^2} \quad \left[\begin{array}{l} u = 1-x^2 \\ \frac{du}{dx} = -2x \\ \frac{du}{-2} = x\,dx \end{array}\right] = \frac{-1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|1-x^2| + C$$

$$\int \frac{dx}{1+x^2} \longrightarrow \text{arc}\tan x$$

what to do?

$$\int \frac{dx}{1-x^2}$$

$$\int \frac{dx}{\sqrt{1+x^2}}$$

TRIG SUBSTITUTIONS

14(8.4) p480

-or- WHAT TO DO WHEN YOU HAVE A SUM OR DIFFERENCE OF SQUARES IN AN INTEGRAL (ESP. A DENOM).

EG 1

$$\int \frac{du}{a^2+u^2} = \frac{1}{a^2}\int \frac{du}{1+\left(\frac{u}{a}\right)^2} \qquad \left[\begin{array}{ll} w = \frac{u}{a} & \text{or } a\,dw = du \\ dw = \frac{du}{a} & \end{array}\right]$$

$$= \frac{a}{a^2}\int \frac{dw}{1+w^2} = \frac{1}{a}\arctan w + C = \frac{1}{a}\arctan\frac{u}{a} + C$$

EG 2 alt. approach: $\int \frac{du}{a^2+u^2}$ Let $u = a\tan\theta \Rightarrow \theta = \arctan\frac{u}{a}$

$du = a\sec^2\theta\, d\theta$

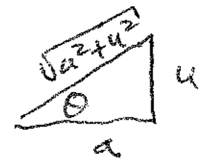
$$= a\int \frac{\sec^2\theta\, d\theta}{a^2+a^2\tan^2\theta} = \frac{a}{a^2}\int \frac{\cancel{\sec^2\theta}\, d\theta}{\cancel{\sec^2\theta}} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C = \frac{1}{a}\arctan\frac{u}{a} + C$$

EG 3 (cf EX1) p481 $\int \frac{du}{\sqrt{a^2+u^2}}$ using same subst as in EG 2

$$= a\int \frac{\sec^2\theta\, d\theta}{\sqrt{a^2\sec^2\theta}} = \frac{\cancel{a}}{\cancel{a}}\int \frac{\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}}\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

if $\tan\theta = \frac{u}{a}$ we have



$\therefore \sec\theta = \frac{\sqrt{a^2+u^2}}{a}$

$$= \ln\left|\frac{\sqrt{a^2+u^2}}{a} + \frac{u}{a}\right| + C = \ln\left|\frac{\sqrt{a^2+u^2}+u}{a}\right| + C$$

$$= \ln\left|\sqrt{a^2+u^2}+u\right| - \ln|a| + C$$

$$= \ln\left|\sqrt{a^2+u^2}+u\right| + C_2$$

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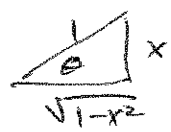
EG 4 We know $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

What about $\int \frac{dx}{1-x^2}$?

Let $x = \sin\theta \quad dx = \cos\theta\, d\theta \quad 1-x^2 = 1-\sin^2\theta = \cos^2\theta$

$\therefore \int \frac{dx}{1-x^2} = \int \frac{\cos\theta\, d\theta}{\cos^2\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

[$x = \sin\theta \Rightarrow$ 

$\Rightarrow \sec\theta = \frac{1}{\sqrt{1-x^2}}$, $\tan\theta \ \frac{x}{\sqrt{1-x^2}}$

$= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C = \ln\left|\sqrt{\frac{(1+x)^2}{1-x^2}}\right| + C$

$= \frac{1}{2}\ln\left|\frac{\cancel{(1+x)}(1+x)}{\cancel{(1+x)}(1-x)}\right| + C = \frac{1}{2}\ln(1+x| - \frac{1}{2}\ln|1-x| + C$

METHOD TO THIS MADNESS

It is important to understand (and remember) the method used in Thms 1, 2, 3 & 4. This method is usually called a trig substitution. We choose which trig function to use based on whether we have a sum or difference and whether the constant is first or not.

$\sin^2\theta + \cos^2\theta = 1 \longrightarrow 1 - \sin^2\theta = \cos^2\theta$ difference – constant first

$1 + \tan^2\theta = \sec^2\theta \longrightarrow 1 + \tan^2\theta = \sec^2\theta$ sum

$\searrow \sec^2\theta - 1 = \tan^2\theta$ difference – constant second

The Form of these trig pythag equations is:

$1 - v^2 = w^2$

$1 + v^2 = w^2$

$v^2 - 1 = w^2$

To simplify, use the substitution:

$v = \sin\theta$

$v = \tan\theta$

$v = \sec\theta$

In general, if you have an expression of the form

$a^2 - u^2$

$a^2 + u^2$

$u^2 - a^2$

$\Longrightarrow$ use the substitution

$u = a\sin\theta$

$u = a\tan\theta$

$u = a\sec\theta$

in order to simplify the expression

Important features to look for:

① plus or minus

② if minus, constant first or second.

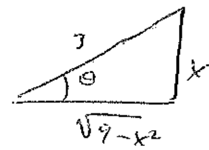
We then choose the substitution appropriately.

7 $\boxed{\text{EX } 8 \text{ p } 476}$

~~8 EX2 p 503~~

~~12~~ 14 EX 3 p 482

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} \qquad \text{Let } x = 3\sin\theta \qquad dx = 3\cos\theta\,d\theta$$



$\sin\theta = \frac{x}{3}$

$$= \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}} = \cancel{27}^{9}\int \frac{\sin^2\theta\,\cos\theta\,d\theta}{\cancel{3}\,\cancel{\cos\theta}}$$

$$= 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int 1-\cos 2\theta\,d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{9}{2}\left[\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$= \frac{9}{2}\left[\arcsin\frac{x}{3} - \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right] + C$$

$$= \frac{9}{2}\left[\arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}\right] + C \quad //$$

12 p470 cf p483 Ex 4 $\int \frac{dx}{\sqrt{25x^2-4}}$

#9

cf ~~p476~~ # 12 cf. $\int \frac{dz}{\sqrt{25z^2-9}}$

$\int \frac{dx}{\sqrt{4x^2-49}}$

minus with variable first indicates a sec substitution.

$25z^2-9$ $\sec^2\theta - 1 = \tan^2\theta$

need a 9

$25z^2-9 = 9\sec^2\theta - 9 = 9\tan^2\theta$

$\Rightarrow 25z^2 = 9\sec^2\theta$

$\Rightarrow 5z = 3\sec\theta$

$\Rightarrow z = \frac{3}{5}\sec\theta$

$\Rightarrow dz = \frac{3}{5}\sec\theta\tan\theta\, d\theta$

In class

$$\int \frac{\frac{3}{5}\sec\theta\tan\theta\, d\theta}{\sqrt{9\tan^2\theta}}$$

$$= \int \frac{\cancel{3}}{5}\frac{\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{3}\,\cancel{\tan\theta}}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta \quad \text{etc.}$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5z + \sqrt{25z^2-9}}{3}\right| + C$$

$\sec\theta = \frac{5z}{3}$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\frac{25z^2}{9} - 1}$

$= \sqrt{\frac{25z^2-9}{9}}$

$= \frac{\sqrt{25z^2-9}}{3}$

8.4

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$$\int \sqrt{8-2x-x^2}\,dx$$

"complete the square" first

$$8-2x-x^2 = 8-(x^2+2x) = 8-(x^2+2x+1-1)$$

$$= 8-(x^2+2x+1)+1 = 9-(x+1)^2 = 3^2-(x+1)^2$$

$$\therefore\ 1+x = 3\sin\theta$$

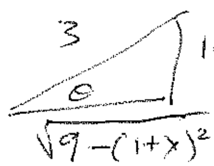
$$dx = 3\cos\theta\, d\theta$$

$$\therefore\ 3^2-(x+1)^2 = 3^2 - 3^2\sin^2\theta = 3^2(1-\sin^2\theta) = 3^2\cos^2\theta$$

$$\therefore \int \sqrt{3^2\cos^2\theta}\cdot 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = \frac{9}{2}\int 1+\cos 2\theta\, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C = \frac{9}{2}\left[\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$1+x = 3\sin\theta \Rightarrow \sin\theta = \frac{1+x}{3} \Rightarrow \theta = \arcsin\frac{1+x}{3}$$



$$\Rightarrow \cos\theta = \frac{\sqrt{8-2x-x^2}}{3}$$

$$= \frac{9}{2}\left[\arcsin\left(\frac{1+x}{3}\right) + \frac{1+x}{3}\,\frac{\sqrt{8-2x-x^2}}{3}\right] + C$$

$$= \frac{9}{2}\arcsin\left(\frac{1+x}{3}\right) + \frac{1}{2}(1+x)\sqrt{8-2x-x^2} + C$$