OVERVIEW OF INTEGRATION TECHNIQUES

Mathematics is the language of patterns

Challenges

1. Find various possible patterns
2. Choose between different patterns
   → which one is best for what I want to do?

Pattern Matching Exercise.

(A) ☆ ☆ ☆ ☆ ☆

(B) □ □ □ □ □

(C) ○ ○ ○ ○ ○

(D) ☆ ☆

(E) □ □ □

(89-20)

TRIG FUNCTIONS (8.3) (~~7.3~~)

7th 4.4 (p 255) Chapt 5 saw integrals of the form

In ~~sect 4.6 (p 292 299)~~ we ~~learned a~~ method

to integrate $\int \sin^n x \cos x \, dx$

7th 4.4 (p 395) 7.1

In sect ~~6.2 (p 408)~~ we learned a method

to integrate $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$.

The Key to these methods was the presence of the derivative of the substitution expression ($d(\sin x) = \cos x$ and $d(\cos x) = -\sin x$). What if the derivative is not present?

ODD POWERS OF SIN (or COS) ALONE

Sample from 1 "case 2" pg 470

p 457 EX

Example of method to use: [~~Same as given in sect 4.4 (pg 238 EX 12)~~]

$$\int \sin^3 x\,dx$$

split $\sin^3 x$ into $\sin^2 x \cdot \sin x$ and transform $\sin^2 x$ into $1-\cos^2 x$

$$= \int (1-\cos^2 x)\sin x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

cf. EX 2 p471

$$= -\int 1-u^2\,du$$

$$= -u + \frac{u^3}{3} + C = \frac{\cos^3 x}{3} - \cos x + C$$

COMMENT: This "expansion" trick can be used for any odd power of $\sin x$ or $\cos x$.

eg.

$$\cos^{\overset{\text{odd \#}}{(2n+1)}} x = (\cos^{2n} x)(\cos x) = (\cos^2 x)^n (\cos x)$$

$$= (1-\sin^2 x)^n \cos x$$

Now let $u = \sin x$
$du = \cos x\,dx$

$$\therefore \int \cos^{2n+1} x\,dx = \int (1-u^2)^n\,du$$

and expand $(1-u^2)^n$ by the binomial theorem.

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EVEN POWERS OF SINES + COSINES (7.4)

p 466

The method used to evaluate integrals of odd powers of sines + cosines can be used to evaluate any integral of the form

$$\int \sin^m x \cos^n x \, dx$$

when at least one of m, n is an odd positive integer.

EX 1
PS 470-71
"Case 1"

E.G. $\int \sin^3 x \cos^2 x \, dx$

$= \int \sin x \sin^2 x \cos^2 x \, dx$

$= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$

$u = \cos x$

$-du = \sin x \, dx$

$= -\int (1 - u^2) u^2 \, du$

$= -\int u^2 - u^4 \, du$

etc

<u>However</u>, <u>if both m and n are even</u>, that approach won't work.

$\therefore$ we must resort to a different method, using the $\frac{1}{2}$ angle formulae.

13th ed. of 5.5 p 343

$$\sin^2 A = \tfrac{1}{2}(1-\cos 2A)$$
$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)$$

"case 3" p 470

7th ed ~~Ex Pg 466~~

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$$

$$= \frac{1}{4}\int (1+\cos 2x)^2\,dx$$

$$= \frac{1}{4}\int 1+2\cos 2x+\cos^2 2x\;dx$$

$$= \frac{1}{4}\int 1+2\cos 2x+\tfrac{1}{2}(1+\cos 4x)\,dx$$

$$= \frac{1}{4}\int \underbrace{1+2\cos 2x+\tfrac{1}{2}}_{3/2}+\tfrac{1}{2}\cos 4x\;dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x+\frac{\cancel{2}\sin 2x}{\cancel{2}}+\frac{1}{2}\,\frac{\sin 4x}{4}\right]+C$$

$$= \frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C \quad //$$

EXAMPLES

Sometimes we need to re-use the method + perform some transformations first.

Ex 3 494   Ex 2 467

EX 3 471

$$\int \sin^2 x \cos^4 x \, dx$$

[1st change completely to all sines or all cosines].

$$= \int (1-\cos^2 x)\cos^4 x \, dx$$

$$= \int \cos^4 x - \cos^6 x \, dx$$

$$= \underbrace{\int \cos^4 x \, dx}_{I_1} - \underbrace{\int \cos^6 x \, dx}_{I_2}$$

$$= I_1 - I_2$$

We know I_1 from previous example.

Let's work on I_2 alone.

$$I_2 = \int \cos^6 x \, dx = \int (\cos^2 x)^3 dx = \int \left[\tfrac{1}{2}(\cos 2x + 1)\right]^3 dx$$

$$= \frac{1}{8}\int 1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x \, dx$$

$$= \frac{1}{8}\int dx + \frac{3}{8}\int \cos 2x \, dx + \underbrace{\frac{3}{8}\int \cos^2 2x \, dx}_{I_3} + \underbrace{\frac{1}{8}\int \cos^3 2x \, dx}_{I_4}$$

$$= \frac{x}{8} + \frac{3}{16}\sin 2x + I_3 + I_4$$

$$I_3 = \frac{3}{16}\int 1 + \cos 4x \, dx = \frac{3}{16}x + \frac{3}{64}\sin 4x + C_3$$

$$I_4 = \frac{1}{8}\int \cos^2 2x \cos 2x \, dx$$

$$= \frac{1}{8}\int (1-\sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x$

$\frac{du}{2} = \cos 2x \, dx$

$$= \frac{1}{8}\cdot\frac{1}{2}\int 1 - u^2 \, du$$

Skip ?

$$= \frac{u}{16} - \frac{u^3}{3\cdot 16} + C_4$$

$$= \frac{\sin 2x}{16} - \frac{\sin^3 2x}{48} + C_4$$

$$\therefore I_2 = \frac{x}{8} + \frac{3}{16}\sin 2x + \underbrace{\frac{3}{16}x + \frac{3}{64}\sin 4x}_{I_3} + \underbrace{\frac{\sin 2x}{16} - \frac{\sin^3 2x}{48}}_{I_4} + C_2$$

$$= \frac{5x}{16} + \frac{\sin 2x}{4} + \frac{3}{64}\sin 4x - \frac{\sin^3 2x}{48} + C_2$$

$$\therefore \int \sin^2 x \cos^4 x \, dx = I_1 - I_2$$

$$= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right] - \left[\frac{5x}{16} + \frac{\sin 2x}{4} + \frac{3\sin 4x}{64} - \frac{\sin^3 2x}{48}\right] + C$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C \quad //$$

OTHER "TRICKS" (square roots)

Sometimes one has to stare for a while before realizing what to do.

8 ~~EX 4 p. 468~~

cf EX 4 p472

$$\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos^2 t}\, dt$$

remember $1-\cos^2 t = \sin^2 t$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\sin^2 t}\, dt = \int_{-\pi/2}^{\pi/2} |\sin t|\, dt$$

$$= \int_{-\pi/2}^{0} |\sin t|\, dt + \int_{0}^{\pi/2} |\sin t|\, dt$$

$$= -\int_{-\pi/2}^{0} \sin t\, dt + \int_{0}^{\pi/2} \sin t\, dt$$

$$= \cos t \Big]_{-\pi/2}^{0} - \cos t \Big]_{0}^{\pi/2} = (1-0) - (0-1) = 2$$

7 ~~Ex 3 467~~

8 ~~EX 4 p475~~

EX 4 p472

$$\int_{0}^{\pi/4} \sqrt{1+\cos 4x}\, dx$$

Note: here there's NO square on cos!

Remember $\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$\therefore \quad 1+\cos 2A = 2\cos^2 A$

$\quad 1+\cos B = 2\cos^2(\frac{B}{2})$

$$\therefore = \int_{0}^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \sqrt{2}\int_{0}^{\pi/4} |\cos 2x|\, dx$$

p ~~460~~ 473 (~~pg 460–1~~)

20-7
7.3

N20-5

PRODUCTS OF TRIG FUNCTIONS

The products of the form

$\sin mx \sin nx$

or $\sin mx \cos nx$

or $\cos mx \cos nx$

Can be either be integrated via <u>parts</u>

or by using trig identities such as

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

4.73
(3)
(~~10~~a)

derived from formulas for the sin + cos of a sum or difference of 2 angles.

pg E/7 p ~~446~~ 473

Using

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

with $m=3$ and $n=5$

we get

$$\sin 3x \cos 5x = \frac{1}{2}[\sin(-2x) + \sin 8x]$$

$$\therefore \int \sin 3x \cos 5x \, dx = \frac{1}{2}\int \sin(-2x) + \sin 8x \, dx$$

$$= \frac{1}{2}\int \sin 8x - \sin 2x \, dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

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HW

pg 467 – 8

\# 6, 20, 31, 33, (35, 37) 40

7, 19,

HW 17

comp pg

7, 9.

SEC^3

12 7版 EX ~~b~~ p ~~459~~ 473

Try parts

$\int \sec^3 x\,dx$ $\qquad u = \sec x \quad dv = \sec^2 x\,dx$

$\qquad\qquad du = \sec x \tan x\,dx \quad v = \tan x$

$$= \sec x \tan x - \int \tan^2 x \sec x\,dx$$
$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$
$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow 2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx = \sec x \tan x + \log|\sec x + \tan x| + C_1$$

$$\Rightarrow \int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\log|\sec x + \tan x| + C_2$$

7版 EX p 4/460

EVEN POWERS OF SEC X and TAN X

12 ~~8版~~ EX5 ~~p~~ 473

a) 7版 cf ~~EX 9 p 257 sec 4.4~~

$$\int \tan^4 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx$$
$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$
$$= \int \underbrace{\tan^2 x}_{u = \tan x} \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx$$
$$= \int u^2\,du - \tan x + x + C$$
$$= \frac{u^3}{3} - \tan x + x + C = \frac{\tan^3 x}{3} - \tan x + x + C \,//$$

b) $$\int \sec^{2n} x\,dx = \int \sec^{2n-2} x \sec^2 x\,dx$$
$$= \int (\sec^2 x)^{n-1} \sec^2 x\,dx = \int (1 + \tan^2 x)^{n-1} \sec^2 x\,dx \qquad u = \tan x,\ du = \sec^2 x\,dx$$
$$= \int (1 + u^2)^{n-1}\,du \quad \underline{etc.}$$

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Not in 12th

$$\int \frac{dx}{\sin^4 x} = \int \csc^4 x\,dx = \int \csc^2 x \csc^2 x\,dx$$

$$= \int \csc^2 x\,(1+\cot^2 x)\,dx$$

Let $u = \cot x$

$du = -\csc^2 x\,dx$

$-du = \csc^2 x\,dx$

$$= -\int 1+u^2\,du$$

$$= -u - \frac{u^3}{3} + C = -\cot x - \frac{\cot^3 x}{3} + C.$$

11, 15, 27

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HW pg 472

1, 4, 14, 16

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5, 4, 14, 16, 21, 37

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