~~9.1~~ 93-10 (B)

NATURAL LOGARITHM (7.1) p ? 420

REVIEW: Given $a^b = c$

then $\log_a c = b$

If $\log_a c = b$ and $\log_a g = f$

then $a^b = c$ and $a^f = g$

Thus $cg = a^b \cdot a^f = a^{(b+f)}$

$\therefore \log_a(cg) = b + f = (\log_a c) + (\log_a g)$

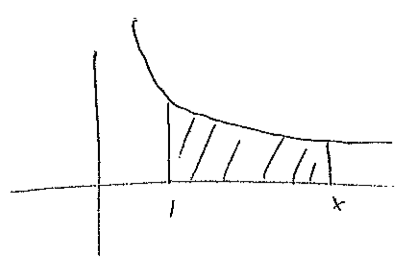
Now, basically 'natural log' is the logarithm to the base $e = 2.7\,1828\,1828\,45\,09\,45\ldots$

We can also define it through an integral

p ~~417~~ 420

$$\log_e x = \ln x = \log x = \int_1^x \frac{1}{t}\,dt$$

Note: most math books use log always = ln for Natural or Napierian log forget about $\log_{10}$.



$\log x$ = area under the curve $\frac{1}{t}$ from 1 to x.

If $x < 1$

$$\log x = \int_1^x \frac{1}{t}\,dx = -\int_x^1 \frac{1}{t}\,dt.$$

Remember

3.3 $\frac{d}{dx}(e^x)$

3.8 $\frac{d}{dx}\ln(x)$

DERIV. OF NAT. LOG.

Since $\log x = \int_1^x \frac{1}{t}\,dt \qquad (x > 0)$

by Thm I Sec. 5.4 (See 4.7)

$$\frac{d}{dx}(\log x) = \frac{1}{x} \qquad \text{cf (3.8)}$$

Then, in general, by the chain rule

$$\frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}$$

NOTE

$$\log 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

Top p ~~416~~ 421

EX

$$y = \log(x^2 + 2x)$$

$$\frac{dy}{dx} = \frac{1}{x^2+2x}\,\frac{d}{dx}(x^2+2x)$$

$$= \frac{1}{x^2+2x}\cdot(2x+2) = \frac{2(x+1)}{x(x+2)} \quad //$$

INTEGRAL FORMULA

$$\int \frac{du}{u} = \log u + C \qquad \text{if } u > 0$$

or

$$\int \frac{du}{u} = \log(-u) + C \qquad \text{if } u < 0$$

p. 423

or

$$\int \frac{du}{u} = \log|u| + C \qquad \text{in general.}$$

NOTE: $\int u^n \, du = \dfrac{u^{n+1}}{n+1} + C$ is true iff $\underline{n \neq -1}$.

Now we have

$$\int u^{-1} \, du = \log|u| + c$$

<u>WARNING</u>: BE CAREFUL NOT TO OVERUSE LOG FORMULA!

$$\int \frac{dx}{x^2} \;\cancel{=}\; \log|x^2| + c$$

$$\hookrightarrow \frac{x^{-1}}{-1} + c$$

EXAMPLES

Not in (3) ~~B 3.6~~ ~~#29.7~~ 7e #27 $\displaystyle\int_{-1}^{0} \frac{dx}{2x+3}$ $\qquad u = 2x+3$, $du = 2\,dx$, $\dfrac{du}{2} = dx$

$$= \frac{1}{2} \int_{x=-1}^{0} \frac{du}{u}$$

$$= \frac{1}{2} \log|u| + c = \frac{1}{2} \log|2x+3| \Big]_{-1}^{0} + c \;// \quad = \frac{1}{2}\log 3 - \frac{1}{2}\cancel{\log 1}^{\,0} = \frac{\log 3}{2}$$

E6 ~~#37~~ $\displaystyle\int \frac{2x-5}{x} \, dx = \int \frac{2x}{x} - \frac{5}{x} \, dx = \int 2\,dx - \int \frac{5\,dx}{x}$

$$= 2\int dx - 5 \int \frac{dx}{x} = 2x - 5\log x + c \;//$$

~~#33~~ 7e 33 $\displaystyle\int \frac{x^2 \, dx}{4-x^3}$ $\qquad u = 4-x^3$, $du = -3x^2\,dx$, $\dfrac{du}{-3} = x^2\,dx$

$$= \frac{1}{-3} \int \frac{du}{u}$$

$$= -\frac{1}{3} \log|u| + c = -\frac{1}{3} \log|4-x^3| + c \;//$$

LOGS AND TRIG FUNCTIONS

~~p 421~~

§5.5

p 344

$$\int \tan x \, dx$$

$$= \int \frac{\sin x \, dx}{\cos x} \qquad \begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \\ -du &= \sin x \, dx \end{aligned}$$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + c$$

$$= -\ln|\cos x| + c \;/\!/ = \ln\left|\frac{1}{\cos x}\right| + c = \ln|\sec x| + c$$

NOTE: $\int \cot x \, dx = \ln|\sin x| + c \;/\!/$

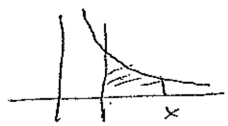
PROPERTIES OF LOGS

p422 cf. sec 1.6 p 45

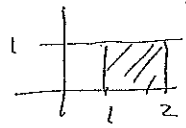
1. $\ln bx = \ln b + \ln x$
2. $\ln \frac{b}{x} = \ln b - \ln x$
3. $\ln \frac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$

THE FUNCTION e^x

Remember, by def., $\ln x = \int_1^x \frac{1}{t}\,dt$

$\Rightarrow$ 

Which value of x gives an area equal to 1?

$x > 2$ since  $\leftarrow$ area = 1

and $\frac{1}{t} < 1$ between $1 < t < 2$

If $x = e = 2.718281828\ldots$, we have

$$1 = \ln e = \int_1^e \frac{1}{t}\,dt$$ p 421

Remember, in gen. $\log_a b = c \Leftrightarrow a^c = b$

If $c = 1$, then $a = b$, i.e. $\log_a a = 1$

$\therefore \ln e = 1 \Rightarrow \ln e = \log_e e$

Note also: $\log_a(a^c) = c$

and $a^c = a^{\log_a b} = b$

$\Rightarrow$ taking logs + evaluating exponents are inverse functions.

$\Rightarrow$ e^x is the inverse of $\ln x$

Sometimes e^x is written $\exp(x)$ (programming languages)

p 424

$\Rightarrow$ $e^{\ln x} = x$

$\ln e^x = x$

DERIVATIVE of E^x

p 425

$$y = e^x \Leftrightarrow x = \log y$$

taking derivatives $\quad 1 = \frac{1}{y}\frac{dy}{dx}$

$$\underbrace{y = \frac{dy}{dx}}\ , = e^x$$

i.e. $\quad y' = y$

or In general, $\quad \frac{d(e^u)}{dx} = e^u \frac{du}{dx}$ $\qquad$ cf (3,3) p 140

and the integral formula.

$$\int e^u \, du = e^u + C.$$

SUBSTITUTION

~~7th pg 449~~ ~~8th ed 484 #15~~

73 $\displaystyle\int_0^{1/2} \frac{e^{\tan^{-1} 2t}}{1+4t^2}\,dt$

Remember if $w = \arctan y$ then $\dfrac{dw}{dy} = \dfrac{1}{1+y^2}$

Let $u = \arctan 2t$

$$du = \frac{2\,dt}{1+4t^2}$$

$$\frac{du}{2} = \frac{dt}{1+4t^2}$$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{e^u}{2} + C = \left.\frac{e^{\arctan 2t}}{2}\right]_0^{1/2} + C \quad //$$

INGENUITY

~~485 #15~~ ~~7 449 #26 #20~~

cf p 428 # 27, 28

$\displaystyle\int 10^{2x}\,dx$

$$y = 10^{2x}$$

$$\log y = 2x \log 10$$

$$\frac{1}{y}\,dy = 2\log 10\,dx$$

$$\frac{1}{10^{2x}}\,dy = 2\log 10\,dx$$

$$\frac{dy}{2\log 10} = 10^{2x}\,dx$$

$$= \frac{1}{2\log 10}\int dy$$

$$= \frac{1}{2\log 10}\,y + C$$

$$= \frac{10^{2x}}{2\log 10} + C \quad //$$

DIFFERENTIAL EQUATIONS

13 ~~12~~ (7.2) p ~~427~~ 430

cf p283 (4.8)

Given an equation of the form

$$\frac{dy}{dx} = f(x)$$

frequently we want to know what function $f(x)$ came from, i.e we want $y = F(x)$ s.t. $F'(x) = f(x) = \frac{dy}{dx}$.

This type of equation is called a <u>differential equation</u> or DIFF - E - Q.

The function $F(x)$ is called the <u>solution</u> of the Diff E.Q. iff $F(x)$ is diff'ble and $\frac{dF(x)}{dx} = f(x)$.

we can also call $F(x)$ the <u>antiderivative</u> or <u>primitive</u> of $f(x)$.

SEPARATING VARIABLES (p ~~428~~)

When solving certain diff eq's, we may have both x's and y's together, e.g.

$$\frac{dy}{dx} = x^2\sqrt{y}$$

Before attempting to solve this by integrating, we must first separate the variables, i.e.

$$\underbrace{\frac{1}{\sqrt{y}}\,dy}_{y\text{'s only}} = \underbrace{x^2\,dx}_{x\text{'s only}}$$

Then we can introduce integrals and perform the integration.

$$\int y^{-1/2}dy = \int x^2 dx \Rightarrow \frac{2y^{1/2}}{1} = \frac{x^3}{3} + C_1 \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C_1$$

$$\Rightarrow y^{1/2} = \frac{x^3}{6} + C_2$$

Ex. With additional information, we can evaluate the constant.

$$\frac{dy}{dx} = x\sqrt{y} \quad \text{when } x=0,\ y=1$$

$$\frac{dy}{\sqrt{y}} = x\,dx \Rightarrow \int y^{-1/2}dy = \int x\,dx \Rightarrow 2y^{1/2} = \frac{x^2}{2} + C$$

$$\Rightarrow 2\cdot 1^{1/2} = \frac{0^2}{2} + C \Rightarrow C = 2$$

$$2y^{1/2} = \frac{x^2}{2} + 2$$

$$4y^{1/2} - x^2 = 4$$

EX2 Solve $y(x+1)\frac{dy}{dx} = x(y^2+1)$.
p 432

Separating variables, we get

$$\frac{y\,dy}{y^2+1} = \frac{x\,dx}{x+1} = \frac{x+1-1}{x+1}dx = 1 - \frac{1}{x+1}dx$$

$$\Rightarrow \int \frac{y\,dy}{y^2+1} = \int 1 - \frac{1}{x+1}\,dx$$

$$\left.\begin{aligned} u &= y^2+1 \\ \frac{du}{dy} &= 2y \\ \frac{du}{2} &= y\,dy \end{aligned}\right\} \Rightarrow \frac{1}{2}\int \frac{du}{u} = \int 1 - \frac{1}{x+1}\,dx$$

$$\Rightarrow \frac{1}{2}\ln|u| = x - \ln|x+1| + C$$

$$\Rightarrow \frac{1}{2}\ln(y^2+1) = x - \ln|x+1| + C$$

EXPONENTIAL CHANGE (7.2)

p ~~428~~ 430

Suppose at time t, a "thing" has measurement $y(t)$.
E.g., The "thing" could be population, temperature difference, amount of radioactive material.

Suppose the measurement/quantity $y(t)$ increases or decreases at a rate k proportional to its size $y(t)$ at a given time t.

Thus, since rate of change is a derivative, we have

$$\frac{dy}{dt} = k \cdot y(t)$$

This is a differential equation.

We may also have an initial condition, eg.

$$y = y_0 \quad \text{when } t = 0$$

To solve, we first separate variables (y-terms on one side, all else on the other). Then we integrate and solve

$$\frac{dy}{y} = k\,dt$$

$$\int \frac{dy}{y} = k\int dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{\ln|y|} = e^{kt+C} = e^{kt} \cdot e^{C}$$

$$\Rightarrow y = \pm e^{C} \cdot e^{kt}$$

Since C is a constant, we can replace $\pm e^{C}$ with another constant, A.

$$\therefore y = Ae^{kt} \quad \text{is the solution}$$

We can evaluate the constant A, using the initial condition, $y = y_0$ when $t = 0$

$$\therefore \quad y = Ae^{kt}$$

$$\Rightarrow \quad y_0 = A e^{0} = A \cdot 1 = A$$

$$\Rightarrow \quad y = y_0 e^{kt},$$

EXAMPLES

Ex #4 p 434 Suppose a model for disease elimination assumes the rate $\frac{dy}{dt}$ (at which # of infected people changes) is proportional to y.

Suppose in any year the number of cases of the disease decreases by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

$\Rightarrow$ we use the eq $y = y_0 e^{kt}$ above.

We want $y = 1000$. Need to find y_0, K, and then t s.t. $y = 1000$.

① y_0: Let "today" correspond to $t=0$. $\therefore$ $y_0 = 10{,}000$.

② K: Since # cases decreases by 20% in a year, at $t = 1$, the # of cases will be 8000.

$\therefore$ we have $\quad 8000 = 10{,}000 \, e^{K \cdot 1} \quad$ from the gen. eq.

$$\Rightarrow \quad e^{K} = .8$$

$$\Rightarrow \quad \ln(e^{K}) = \ln(0.8)$$

$$\Rightarrow \quad K = \ln(0.8)$$

Thus, the gen eq for this problem is now

$$y = 10{,}000\, e^{(\ln 0.8)t}$$

(3) t s.t. $y = 1000$

Using the gen eq for this problem, we have

$$1000 = 10{,}000\, e^{(\ln 0.8)t}$$

$$\Rightarrow \quad e^{(\ln 0.8)t} = .1$$

$$\ln(0.8)t = \ln 0.1$$

$$\Rightarrow \quad t = \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years.}$$

Thus it would take about 10 years and 4 months to reduce the # cases from 10,000 to 1000 under our assumptions.