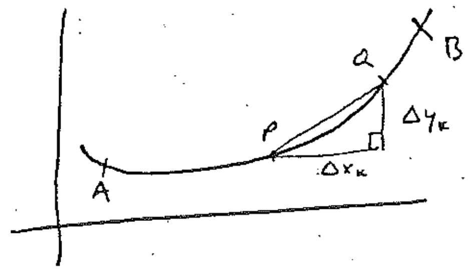
LENGTH OF PLANE CURVE (S/S) (6.3)

Look at a curve and pick 2 "end points" A and B



[Think]

Pick 2 other points, P and Q.

The straight line distance from P to Q is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$

This straight line distance between P and Q <u>approximates</u> the distance along the curve between P and Q.

To find the approximate distance between A and B on the curve, we can pick a number of points between A and B, calculate the straight line distances between them, then add these distances together to find the approximate distance.

I.e.

$$\text{Length from A to B} = L_A^B \approx \sum^{n} \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

If we increase the number of points between A and B, we also decrease the length and make the approximate distance closer to the exact distance along the curve. As $n \to \infty$, the approximate distance becomes exact.

i.e.

$$L_A^B = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

Let's work with the right side

If the curve $y = f(x)$ is continuous and has a continuous derivative at each point from A to B, then by the Mean Value Thm,

$\exists\, c_k$ s.t. $\quad \Delta y_k = f'(c_k)\Delta x_k$

Therefore

$$L_A^B = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + \left(f'(c_k)\Delta x_k\right)^2}$$

$$= \lim_{\Delta x\to 0} \sum_{a}^{b} \sqrt{\left(1 + \left(f'(c_k)\right)^2\right)\Delta x_k^2}$$

$$= \lim_{\Delta x\to 0} \sum_{a}^{b} \sqrt{1 + \left(f'(c_k)\right)^2}\ \Delta x$$

$$= \int_a^b \sqrt{1 + f'(\cancel{x})^2}\, dx \qquad (3)$$ Box 383

$$= \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{or} = \int_{y=c}^{y=d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (4)$$ 384

or if x and y are given parametrically

$$= \int_{t=t_a}^{t=t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (3)$$

$$\text{or} = \int \sqrt{dx^2 + dy^2} = \int ds \qquad (7)$$

Box on pg 384

Chapter 11

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EXAMPLES

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#2 Length of curve $y = x^{3/2}$ from $(0,0)$ to $(4,8)$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

$$L_A^B = \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^4 \sqrt{1+\left(\tfrac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$= \int_0^4 \sqrt{1+\frac{9x}{4}}\,dx \qquad u = 1+\frac{9}{4}x$$

$$du = \frac{9}{4}dx$$

$$= \frac{4}{9}\int_{x=0}^{x=4} u^{\frac{1}{2}}\,du \qquad \frac{4}{9}du = dx$$

$$= \frac{4}{9}\left[\frac{2u^{3/2}}{3}\right]_{x=0}^{x=4}$$

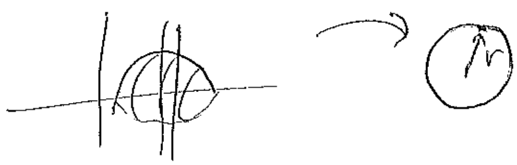
$$= \frac{8}{27}\left[\left(1+\frac{9}{4}x\right)^{3/2}\right]_{x=0}^{x=4}$$

$$= \frac{8}{27}\left[(1+9)^{3/2} - 1^{3/2}\right] = \frac{8}{27}\left[10\sqrt{10}-1\right]$$

VOLUME – SUMMARY

Rotation around line (axis)

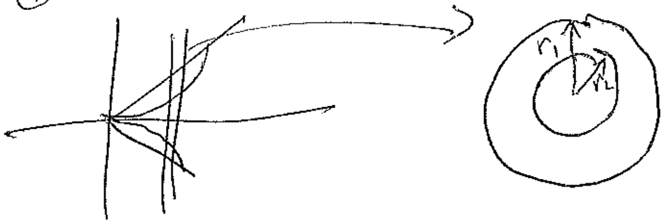
Solid centre



$$\int_a^b \pi r^2 \, dx = \int_a^b \pi y^2 \, dx$$

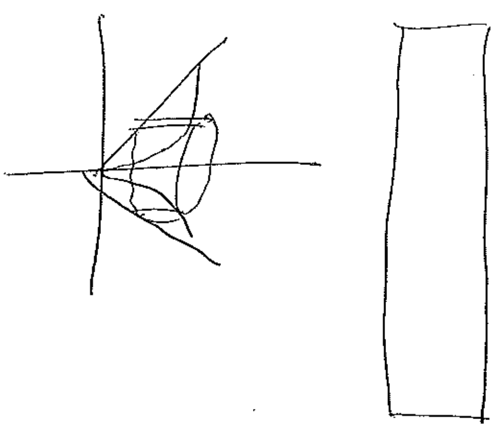
Hollow centre

① washer



$$\int_a^b \pi \left(r_1^2 - r_2^2\right) dx$$

② cylindrical shells



$$\int_a^b 2\pi r \cdot h \quad dy$$

r – measured along differential axis y

h – measured along other axis (x)

328 344

\# 11 17

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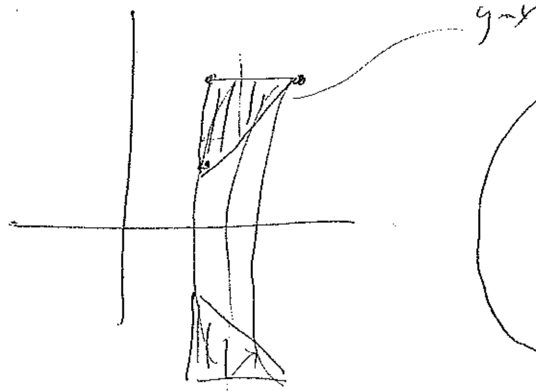
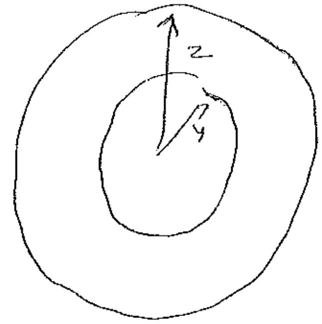
(1, 1)

(1, 2)

(2, 2)





ⓐ x-axis

washers

$$\int_1^2 \pi 2^2 - \pi y^2 \, dx = \pi \int_1^2 4 - x^2 \, dx$$

$$= \pi \left[4x - \frac{x^3}{3} \right]_1^2 = \pi \left[8 - \frac{8}{3} \right] - \pi \left[4 - \frac{1}{3} \right]$$

$$= \pi \left[\frac{16}{3} - \frac{11}{3} \right] = \underline{\underline{\frac{5\pi}{3}}}$$

$r = y$
$h = x - 1$

ⓑ shells.

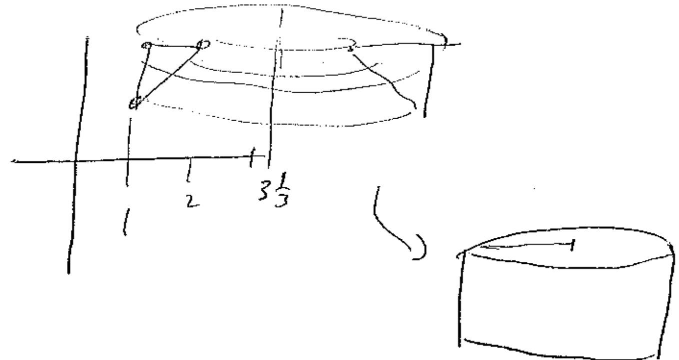
$$\int_1^2 2\pi r h \, dy = 2\pi \int_1^2 y \left(x - 1 \right) dy$$

$$= 2\pi \int_1^2 y (y - 1) \, dy = 2\pi \int_1^2 y^2 - y \, dy$$

$$= 2\pi \left[\frac{y^3}{3} - \frac{y^2}{2} \right]_1^2 = 2\pi \left[\left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right]$$

$$= 2\pi \left[\frac{2}{3} - \left(\frac{2}{6} - \frac{3}{6} \right) \right] = 2\pi \left[\frac{4}{6} - \left(-\frac{1}{6} \right) \right] = 2\pi \frac{5}{6} = \underline{\underline{\frac{5\pi}{3}}}$$

(c) $\text{line } x = \frac{10}{3} = 3\tfrac{1}{3}$



min rad $= 3\tfrac{1}{3} - 2 = 1\tfrac{1}{3} = \frac{4}{3}$

max rad $= 3\tfrac{1}{3} - 1 = 2\tfrac{1}{3} = \frac{7}{3}$

gen rad $= 3\tfrac{1}{3} - x \qquad 1 \le x \le 2$

gen height $= \cancel{y - 1 = x - 1}$
$2 - y = 2 - x$

$$\int_1^2 2\pi r h \, dx = 2\pi \int_1^2 \left(3\tfrac{1}{3} - x\right)(2 - x)\, dx$$

$$= 2\pi \int_1^2 \frac{20}{3} - 2x - \frac{10}{3}x + x^2 \, dx$$

$$= 2\pi \int_1^2 \frac{20}{3} - \frac{16}{3}x + x^2 \, dx$$

$$= 2\pi \left[\frac{20}{3}x - \frac{\cancel{16}^{\,8}}{3}\frac{x^2}{\cancel{2}} + \frac{x^3}{3} \right]_1^2$$

$$= 2\pi \left[\frac{20}{3} \cdot 2 - \frac{8}{3} 4 + \frac{8}{3} - \left(\frac{20}{3} - \frac{8}{3} + \frac{1}{3} \right) \right]$$

$$= \frac{2\pi}{3} \left[40 - 32 + 8 - 20 + 8 - 1 \right]$$

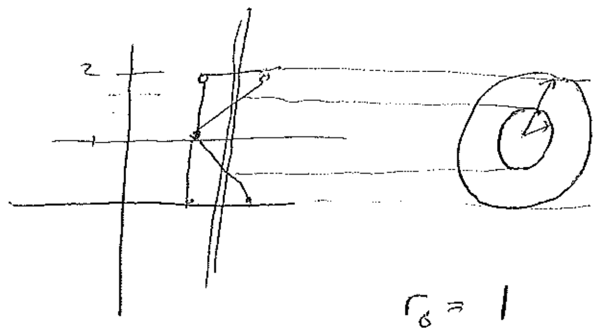
$$= \frac{2\pi}{3} \left[56 - 53 \right] = \frac{2\pi}{3} \cdot 3 = 2\pi$$

[illegible] cd

5.3 #17

(d) rotated around $y = 1$

washer



$r_o = 1$

$r_i = y - 1 = x - 1$

$$\int_1^2 \pi r_o^2 - \pi r_i^2 \, dx = \pi \int_1^2 1 - (x-1)^2 \, dx$$

$$= \pi \int_1^2 1 - (x^2 - 2x + 1) \, dx = \pi \int_1^2 \cancel{1} - x^2 + 2x - \cancel{1} \, dx$$

$$= \pi \int_1^2 2x - x^2 \, dx = \pi \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3} \right]_1^2$$

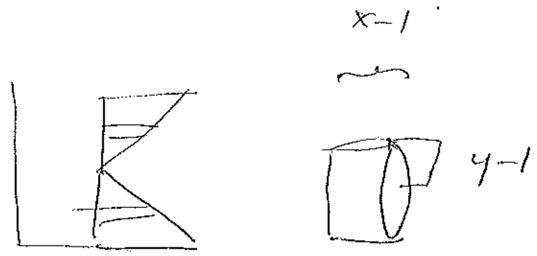
$$= \pi \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) \right] = \pi \left[4 - 1 - \frac{8}{3} + \frac{1}{3} \right]$$

$$= \pi \left[3 - \frac{7}{3} \right] = \pi \left[\frac{9-7}{3} \right] = \frac{2\pi}{3}$$

8th ed.

1d)

shells



$$\int 2\pi r h \, dy = 2\pi \int_1^2 (y-1)(y-1)\, dy$$

$$2\pi \int_1^2 (y-1)^2 dy = \frac{2\pi (y-1)^3}{3} \Big]_1^2$$

$$\frac{2\pi}{3}\left[(2-1)^3 - (1-1)^3\right] = \frac{2\pi}{3}$$

EXAMPLE (HW?)

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practice prob

p 416 # 16

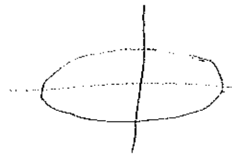
p 414

$a = \frac{11}{2}$

$b = 2\sqrt{3}$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$y = 0 \Rightarrow x = \pm a$



$$y^2 = b^2 - \frac{b^2x^2}{a^2}$$

$$V = 2\int_0^a \pi y^2 \, dx$$

$$= 2\pi \int_0^a \left(b^2 - \frac{b^2x^2}{a^2} \right) dx$$

$$= 2\pi \left[b^2x - \frac{b^2x^3}{3a^2} \right]_0^a$$

$$= 2\pi \left[ab^2 - \frac{\cancel{a^3}b^2}{3\cancel{a^2}} \right]$$

$$= 2\pi \left[ab^2 - \frac{1}{3}ab^2 \right]$$

$$= 2\pi \left[\frac{2}{3}ab^2 \right] = \frac{4\pi}{3}ab^2$$

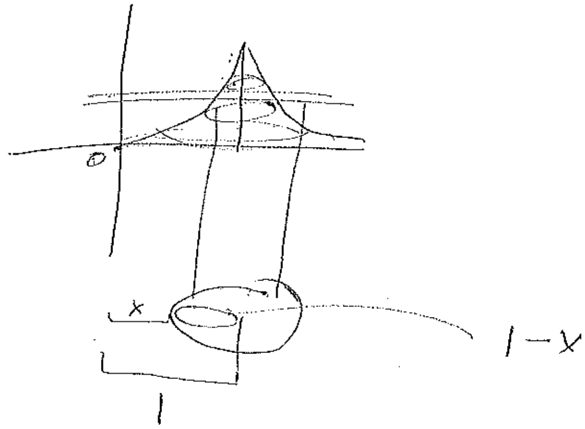
7

~~319 #22~~

12?

$$\left[\begin{array}{l} y = x^{3/2} \\ x\text{-axis} \\ x = 1 \end{array}\right] \text{area}$$

rotate and $x=1$



$$\int_0^1 \pi (1-x)^2\, dy$$

$$y^{2/3} = x$$

$$\int_0^1 \pi (1-y^{2/3})^2\, dy$$

$$\pi \int_0^1 1 - 2y^{2/3} + y^{4/3}\, dy$$

$$\pi \left[y - \frac{3\cdot 2y^{5/3}}{5} + \frac{3y^{7/3}}{7} \right]_0^1$$

$$\pi \left[1 - \frac{6}{5}\cdot 1 + \frac{3}{7}\cdot 1 \right] - 0$$

$$\pi \left[\cancel{} \right] = -\frac{\pi}{14}$$

$$\pi \left[\frac{35 - 42 + 15}{35} \right] = \pi \frac{[50-42]}{35} = \underline{\underline{\frac{8\pi}{35}}}$$