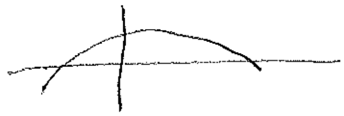
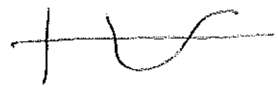
ROLLE'S THEOREM (4.2) p 230

Look at a "smooth" curve (i.e. diff'ble)
Suppose it crosses the axis in 2 places.
i.e.

 or 

Then we can deduce that there is a point between the 2 x-intercepts where the tangent is flat (i.e. horizontal)
∴ The curve has a max or min and
$f'(x) = 0$ at that point.

This is basically <u>ROLLE'S Thm</u>.
more formally.

ROLLE'S THM Thm 3 p 230

Let f be a function, defined, cnts on closed interval $[a,b]$.
Let f be diff'ble on an open interv. (a,b).
Let $f(a) = f(b) \boxed{= 0}$
Then $\exists$ at least 1 point c, $a < c < b$ s.t.

$$f'(c) = 0.$$

proof.
Case I $f(x) = \overset{f(a)=f(b)}{\cancel{0}}$ $\forall x \in [a,b]$
$\therefore f'(x) = 0 \quad \forall x \in [a,b]$ //

case II If $f(x) \neq 0 \quad \forall x \in [a,b]$
$\exists\, x$ s.t. $f(x) > 0$ or $f(x) < 0$
∴ $\exists$ some max pos value or min neg val.
Let $f(c)$ be that max/min value
∴ by Thm 2. (4.1) p 225, $f'(c) = 0$ //

APPLICATION

EX 1
p 230

show $x^3+3x+1=0$ has exactly 1 real solution

Let $f(x) = x^3+3x+1$

Pick 2 pts x_1, x_2 s.t. $f(x_1) < 0$ and $f(x_2) > 0$

e.g. $x_1 = -1$ ($\because f(-1) = -3$) and $x_2 = 0$ ($\because f(0) = 1$)

Since $f(x)$ is cont's, by Intermediate Value Thm (2.5, p99)

there must be a point where $f(x) = 0$, and x is between -1 and 0.

Note that $f'(x) = 3x^2+3 \geq 3$

If there were a second point b where $f(b) = 0$, then by Rolle's Thm, there MUST be a point where $f'(x) = 0$ but that is impossible

$\therefore$ Second point b s.t. $f(b)=0$ can not exist!

GENERALIZING ROLLE'S THM

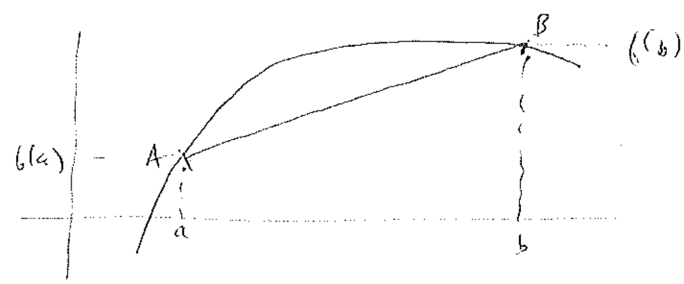
p 231

If the slope of $f(x)$ is 0 at a point c, this means the tangent is parallel to the x-axis at that point.

Let's try to generalize this to other lines, not only the x-axis.

Specifically, pick 2 points on a curve, A, B.

$A = (a, f(a))$
$B = (b, f(b))$



The slope of the chord AB $= \dfrac{f(b)-f(a)}{b-a}$

We would like to prove that $\exists\, c$ between a and b st. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ or $f(b)-f(a) = f'(c)(b-a)$

i.e. the line tangent to $f(x)$ at c is parallel to the chord AB.

This is the <u>Mean Value Thm</u>.

MEAN VALUE THM

Note: "Mean" means "average"

Thm 4 (MVT) 12.23

Let $y = f(x)$ be cnts for $x \in [a,b]$.
Let y be diff'ble for $x \in (a,b)$.
Then $\exists$ at least 1 number $c \in (a,b)$
s.t. $f(b) - f(a) = f'(c)(b-a)$

proof.

Let $g(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$

this is a straight line through pts A + B.

$g(a) = f(a)$
$g(b) = f(b)$

$d = f - g$
$h \quad h = f - g$

Let $h(x) = f(x) - g(x)$

Then $h(a) = f(a) - f(a) = 0$
and $h(b) = f(b) - f(b) = 0$

$\therefore h$ satisfies Rolle's Thm.

$\therefore \exists\ c$ s.t. $h'(c) = 0$ for $c \in (a,b)$

$$h'(x) = f'(x) - g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

$$\therefore h'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} = 0$$

$$\therefore f'(c) = \frac{f(b) - f(a)}{b-a}$$

or $f(b) - f(a) = f'(c)(b-a)$ // QED

EXAMPLE

Given $f(x) = x^{2/3}$ find c of the MVT.
given $a = 0$, $b = 1$

$f(a) = f(0) = 0$ $\quad$ $A = (0,0)$

$f(b) = f(1) = 1$ $\quad$ $B = (1,1)$

$\therefore$ slope of $AB = \dfrac{f(b) - f(a)}{b - a} = \dfrac{1}{1} = 1$

We want c s.t. $f'(c) = 1$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{2}{3c^{1/3}} = 1$$

$$\therefore \frac{2}{3} = c^{\frac{1}{3}}$$

$$\text{or } \underline{\underline{c = \frac{8}{27}}}$$

Corol 1

If $F'(x) = 0 \quad \forall x \in (a,b)$

then $F(x) = \text{constant} \quad \forall x \in (a,b)$

Proof

pick x_1, x_2 st. $a < x_1 < x_2 < b$

By MV Thm $\exists\, c$ s.t. $F(x_1) - F(x_2) = (x_1 - x_2) F'(c)$

But we assume $F'(c) = 0$

$\therefore \quad F(x_1) = F(x_2) \quad \forall x_1, x_2 \in (a,b)$

i.e. $F(x) = \text{constant}$.

Corol 2

Let F_1, F_2 be 2 functions

st. $F_1'(x) = F_2'(x) = f(x)$ for $x \in (a,b)$

then $F_1(x) - F_2(x)$ is a constant.

Proof

Let $G(x) = F_1(x) - F_2(x)$

And apply Corol 1 to $G(x)$ //

ACCELERATION, VELOCITY AND LOCATION (p 234)

In 3.4 (p 145), we learned, if s is location,
$v = \frac{ds}{dt}$ (velocity) and $a = \frac{dv}{dt}$ (acceleration)

Also we saw free-fall metric equation

$$s = 4.9t^2 \quad \text{and} \quad v(t) = 9.8t$$

or in feet,

$$s = 16t^2 \quad \text{and} \quad v(t) = 32t$$

With additional information (and cord. 2) we can determine some vel + loc equations.

EG.

p 237 #44

$$v = 32t - 2 \qquad s(0.5) = 4$$

$$s(t) = 16t^2 - 2t + C$$

$$s(\tfrac{1}{2}) = 4 = 16(\tfrac{1}{2})^2 - 2(\tfrac{1}{2}) + C$$

$$= \frac{16}{4} - 1 + C$$

$$= 4 - 1 + C$$

$$\Rightarrow 4 = 3 + C \Rightarrow C = 1$$

$$\therefore \; s(t) = 16t^2 - 2t + 1$$