Math 12-6 (10:30 AM)—Spring, 2025
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Final (**Two** hours)
Monday, June 9, 2025, 9:10 AM

\# perfect scores per problem

6/14
1. (10) *(Taken from Midterm I.)* Evaluate $\int \cos^2 2x\, dx$.

5/14
2. (10) *(Taken from Midterm II.)* You are given the area bounded by the curve $y = \sec 2x$, the x-axis, and the lines $x = 0$ and $x = \frac{\pi}{8}$. Rotate this area around the x-axis and find the volume of the resulting solid.

5/14
3. (10) *(Taken from Midterm III.)* Evaluate: $\int x^3 e^{x^2} dx$

7/14 EASIEST TIE
4. (10) *(Taken from the Fall 2023 sample final.)* Evaluate $\int \frac{dx}{1-x^2}$ via trig substitution.

4/14
5. (12) Solve $\frac{dy}{dx} = e^{-2x}y$ given $x = 0$ and $y = 1$.

5/14
6. (12) Perform the following integration by means of partial fractions, WITHOUT evaluating the coefficients, A, B, ...: $\int \frac{dx}{(x^2-4)(x^2+1)}$. (I.e., you do NOT have to compute numerical values for A, B, ..., but merely include these coefficients in the final answer.)

7/14 EASIEST TIE
7. (12) Integrate $\int x^3\sqrt{x^2-4}\, dx$.

5/14
8. (13) Evaluate $\int_{-1}^{1} \frac{1}{x^4} dx$.

1/14 HARDEST
9. (36) *Examine*, but DO NOT *integrate* the following 6 expressions. By *examine*, please indicate the NUMBER of the method you would use to perform the integration from the possible methods listed below. *If more than one method is possible, indicate the "best" (i.e., the simplest) method.* NOTE: you do NOT have to use a different method for each integral — this problem could be arranged so that all integrals might be done by one and the same method (unlikely though that be)!

(a) $\int \frac{1}{x^2-x^3}\, dx$ (b) $\int \frac{1}{\sqrt{x^2-2x+1}}\, dx$ (c) $\int \ln x\, dx$

(d) $\int \frac{1}{\sqrt{1-x^2}}\, dx$ (e) $\int \frac{x}{\sqrt{1-x^2}}\, dx$ (f) $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$

\# 1. SIN substitution. # 2. TAN substitution. # 3. SEC substitution.
\# 4. partial fractions (after first factoring, if necessary).
\# 5. complete the square, then a trig substitution (<u>indicate which one</u>).
\# 6. simple substitution (<u>indicate what u equals</u>) and the rule after the substitution.
\# 7. integration by parts (including the "table" version). <u>Indicate u and dv</u> (or $f(x)$ and $g(x)$).
\# 8. algebraically simplify expression first, then use an elementary rule NOT listed (<u>indicate which rule</u>).

125 points total.

P.S. Have a great summer!

STATS HI 111/125 MED ⟨95⟩ σ 19.72
LO 57 MEAN 90.79

MT 12 − S 25 − FINAL

1. $\int \cos^2 2x\,dx = \frac{1}{2}\int 1+\cos 4x\,dx = \frac{1}{2}x + \frac{1}{2}\frac{\sin 4x}{4} + C = \frac{x}{2} + \frac{\sin 4x}{8} + C$

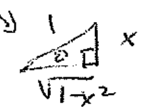
2. $V = \pi\int_0^{\pi/8}(\sec 2x)^2\,dx = \begin{bmatrix} u = 2x \\ \frac{du}{dx} = 2 \\ \frac{du}{2} = dx \end{bmatrix} \frac{\pi}{2}\int_{x=0}^{\pi/8}\sec^2 u\,du = \frac{\pi}{2}\tan u\Big]_{x=0}^{\pi/8}$

$= \frac{\pi}{2}\tan 2x\Big]_0^{\pi/8} = \frac{\pi}{2}\left[\tan 2\left(\frac{\pi}{8}\right) - \tan 0\right] = \frac{\pi}{2}\tan\frac{\pi}{4} = \frac{\pi}{2}$

3. $\int x^3 e^{x^2}\,dx = \int x^2 e^{x^2} x\,dx \quad \begin{bmatrix} w = x^2 & \frac{dw}{2} = x\,dx \\ \frac{dw}{dx} = 2x & \end{bmatrix} = \frac{1}{2}\int w e^w\,dw \quad \begin{bmatrix} u = w & dv = e^w\,dw \\ du = dw & v = e^w \end{bmatrix}$

$= \frac{1}{2}\left[we^w - \int e^w\,dw\right] = \frac{1}{2}we^w - \frac{1}{2}e^w + C = \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

4. $\int \frac{dx}{1-x^2} \quad \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} = \int \frac{\cos\theta\,d\theta}{1-\sin^2\theta} = \int \frac{\cos\theta}{\cos^2\theta}\,d\theta = \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C$ [] $= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$

5. $\frac{dy}{dx} = e^{-2x}y,\ x=0,\ y=1 \Rightarrow \frac{dy}{y} = e^{-2x}\,dx \Rightarrow \int\frac{dy}{y} = \int e^{-2x}\,dx \Rightarrow \ln|y| = \frac{-e^{-2x}}{2} + C$

$\Rightarrow \ln 1 = -\frac{1}{2}e^0 + C \Rightarrow 0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2} \Rightarrow \ln|y| = \frac{-e^{-2x}}{2} + \frac{1}{2}$

6. $\int \frac{dx}{(x^2-4)(x^2+1)} \quad \left[\frac{1}{(x^2-4)(x^2+1)} = \frac{1}{(x-2)(x+2)(x^2+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x^2+1} + \frac{D(2x)}{x^2+1}\right]$

$= A\int\frac{dx}{x-2} + B\int\frac{dx}{x+2} + C\int\frac{dx}{x^2+1} + D\int\frac{2x\,dx}{x^2+1} \quad \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix}$

$= A\ln|x-2| + B\ln|x+2| + C\arctan x + D\int\frac{du}{u}$

$= A\ln|x-2| + B\ln|x+2| + C\arctan x + D\ln|x^2+1| + C_2$

7. $\int x^3\sqrt{x^2-4}\,dx$ $\left[\begin{array}{l} u+4=x^2 \\ u=x^2-4 \\ du=2x\,dx \\ \frac{du}{2}=x\,dx \end{array}\right] = \int x^2\sqrt{x^2-4}\;x\,dx = \frac{1}{2}\int (u+4)\,u^{1/2}\,du$

$= \frac{1}{2}\int u^{3/2}+4u^{1/2}\,du = \frac{1}{\cancel{2}}\,\frac{u^{5/2}}{5/\cancel{2}} + \frac{4}{\cancel{2}}\,\frac{u^{3/2}}{3/\cancel{2}} = \underline{\frac{(x^2-4)^{5/2}}{5} + \frac{4(x^2-4)^{3/2}}{3} + C}$

8. $\int_{-1}^{1} \frac{1}{x^4}\,dx$ improper integral at $x=0$

Look at $\lim_{b\to 0^+}\int_b^1 x^{-4}\,dx = \lim \left.\frac{x^{-3}}{-3}\right]_b^1 = \lim\left(-\frac{1}{3}-\left(-\frac{1}{3b^3}\right)\right) = \lim \frac{1}{3}\left(\frac{1}{b^3}-1\right) = \infty$

$\therefore$ entire integral diverges

9. a) #4 partial fractions

b) #8 algebraic simplification

$\sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = x-1$

Then use $\int \frac{du}{u} = \ln|u|+C = \ln|x+1|+C$

c) #7 parts

$u=\ln x \quad dV=dx$

d) #1 sin subst

or

#8 arcsin formula

e) #6 simple subst.

$u=1-x^2$

+ power rule

f) #1 sin subst.