Math 12-7 (1:00 PM)—Fall, 2023 | Final (2 hours)
D.C. Smolarski, S.J. | Wednesday, December 6, 2023, 1:30 PM

perfect scores per problem

24/34 1. (10) *(Taken from Midterm I.)* Integrate $\int \sin^2 3x\,dx$.

27/34 EASIEST 2. (10) *(Taken from Midterm II.)* Evaluate $\int_2^3 \frac{(1-x^2)^2}{x}dx$

24/34 3. (10) *(Taken from Midterm III.)* The area between the lines $y = e^{2x}$, $x = 1$, and $x = 2$ is rotated around the x-axis. Find the volume of the resulting solid.

23/34 4. (10) *(Taken from the Winter 2023 sample final.)* Integrate $\int x^3\sqrt{x^2-4}\,dx$.

12/34 HARDEST 5. (12) Perform the following integration by means of partial fractions, WITHOUT evaluating the coefficients, A, B, ...: $\int \frac{dx}{(x^2+x)(4x^2+4)}$. (I.e., you do NOT have to compute numerical values for A, B, ..., but merely include these variables in the final answer.)

13/34 6. (13) Evaluate $\int_{-2}^3 \frac{x^2+x^4}{x^6}dx$.

19/34 7. (12) Integrate $\int \frac{dx}{1-x^2}$ via trig substitution.

25/34 8. (12) Integrate $\int x^2 \ln x\,dx$.

14/34 9. (36) Examine, but DO NOT integrate the following 6 expressions. By *examine*, please indicate the NUMBER of the method you would use to perform the integration from the possible methods listed below. *If more than one method is possible, indicate the "best" (i.e., the simplest) method.* NOTE: you do NOT have to use a different method for each integral — this problem could be arranged so that all integrals might be done by one and the same method (unlikely though that be)!

(a) $\int \frac{1}{\sqrt{x^2-1}}\,dx$ (b) $\int \frac{1}{x^2-1}\,dx$ (c) $\int \frac{x}{\sqrt{x^2-1}}\,dx$

(d) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (e) $\int xe^x dx$ (f) $\int \frac{\ln x}{x}\,dx$

1. SIN substitution. # 2. TAN substitution. # 3. SEC substitution.
4. partial fractions (after first factoring, if necessary).
5. complete the square, then a trig substitution (indicate which one).
6. simple substitution (indicate what u equals).
7. integration by parts (including the "table" version). Indicate u and dv (or $f(x)$ and $g(x)$).
8. algebraically simplify expression first (if needed), then use an elementary rule NOT listed (indicate which rule).

125 points total. P.S. Have a restful Christmas break!

STATS

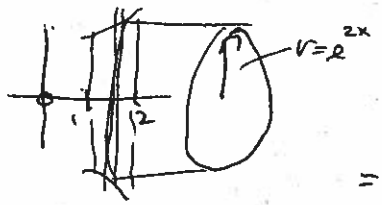
HI 125/125 MEDIAN <111.5> # EXAMS 34
LO 43 MEAN 99.882

Mt 12 – FINAL – F23

1. $\int \sin^2 3x\,dx = \frac{1}{2}\int 1-\cos 6x\,dx = \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + C$

2. $\int_2^3 \frac{(1-x^2)^2}{x}\,dx = \int \frac{1-2x^2+x^4}{x}\,dx = \int \frac{1}{x} - 2x + x^3\,dx = \ln x - \frac{2x^2}{2} + \frac{x^4}{4}\Big]_2^3$

$= \ln 3 - 9 + \frac{81}{4} - \ln 2 + 4 - 16 = \ln\frac{3}{2} + \frac{81}{4} - \frac{36}{4} = \ln\frac{3}{2} + \frac{45}{4}$

3. 

$V = \pi\int_1^2 (e^{2x})^2\,dx = \pi\int e^{4x}\,dx \quad \left[\begin{matrix} u=4x \\ \frac{du}{4}=dx \end{matrix}\right] = \frac{\pi}{4}\int e^u\,du$

$= \frac{\pi}{4}e^u\Big]_{x=1}^{2} = \frac{\pi}{4}e^{4x}\Big]_1^2 = \frac{\pi}{4}\left(e^8 - e^4\right)$

4. $\int x^3\sqrt{x^2-4}\,dx \quad \left[\begin{matrix} u = x^2-4 \\ du = 2x\,dx \\ \frac{du}{2} = x\,dx \\ u+4 = x^2 \end{matrix}\right] = \int x^2\sqrt{x^2-4}\,x\,dx = \frac{1}{2}\int (u+4)u^{1/2}\,du$

$= \frac{1}{2}\int u^{3/2} + 4u^{1/2}\,du = \frac{1}{2}\frac{u^{5/2}}{5/2} + \frac{4}{2}\frac{u^{3/2}}{3/2} + C = \frac{(x^2-4)^{5/2}}{5} + \frac{4(x^2-4)^{3/2}}{3} + C$

5. $\frac{1}{(x^3+x)(4x^2+4)} = \frac{1}{x(x+1)\,4(x^2+1)} = \frac{A}{4x} + \frac{B}{x+1} + \frac{C}{x^2+1} + \frac{D\,2x}{x^2+1}$

$\Rightarrow \int \frac{dx}{(x^3+x)(4x^2+4)} = \frac{A}{4}\int\frac{dx}{x} + B\int\frac{dx}{x+1} + C\int\frac{dx}{x^2+1} + D\int\frac{2x\,dx}{x^2+1}$

$= \frac{A}{4}\ln|x| + B\ln|x+1| + C\arctan x + D\ln(x^2+1) + C_2$

6. $\int_{-2}^{3} \frac{x^2+x^4}{x^6}\,dx = \int_{-2}^{3} \frac{1}{x^4} + \frac{1}{x^2}\,dx$ ← improper integral at $x=0$

$\lim_{b\to 0^-}\int_{-2}^{b} x^{-4} + x^{-2}\,dx = \lim \frac{x^{-3}}{-3} + \frac{x^{-1}}{-1}\Big|_{-2}^{b} = \lim\left(\frac{-1}{3x^3} - \frac{1}{x}\right)\Big|_{-2}^{b}$

$= \lim\left(\underbrace{\frac{-1}{3b^3} - \frac{1}{b}}_{\text{goes to } +\infty \text{ as } b\to 0^-} + \frac{1}{3(-2)^3} + \frac{1}{-2}\right)$

$\therefore$ integral diverges

7. $\int \frac{dx}{1-x^2}$ $\left[\begin{matrix} x=\sin\theta \\ dx=\cos\theta\, d\theta \end{matrix}\right] \Rightarrow \int \frac{\cos\theta\, d\theta}{1-\sin^2\theta} = \int \frac{\cos\theta\, d\theta}{\cos^2\theta} = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$ $\left[\begin{matrix} \text{hyp } 1,\ \theta,\ x,\ \sqrt{1-x^2} \end{matrix}\right] = \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$

8. $\int x^2 \ln x\, dx$ $\left[\begin{matrix} u=\ln x & dv=x^2 dx \\ du=\frac{dx}{x} & v=\frac{x^3}{3} \end{matrix}\right]$

$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int \frac{x^3}{x}\, dx$

$= \frac{x^3 \ln x}{3} - \frac{1}{3}\frac{x^3}{3} + C = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$

9. a) #3 sec sub

b) #3 sec
or #4 partial fract

c) #6 $u = x^2 - 1$

d) #1 sin. subst
or #8 are $\sin x$

e) #7 parts
$u = x$
$dv = e^x dx$

f) #6 $u = \ln x$