1.1 Functions + graphs

Domain x, Range y $\quad y=x^2 \quad D\ (-\infty,\infty) \quad R\ [0,\infty)$

$y=\sqrt{x} \quad D\ (0,\infty) \quad R\ [0,\infty)$

Function satisfy "Vertical line test"

Piecewise defined eg. $y=|x| = \begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$

Special function $y=\lfloor x \rfloor$ greatest integer

$= \lceil x \rceil$ ceiling function.

Odd function $f(-x) = -f(x)$ eg $y=x^3$ ← sym about origin

Even function $f(-x) = f(x)$ eg $y=x^2$ ← sym about y-axis.

1.2 Combining functions.

Sum, diff, products, quotients

→ No Problem, except avoid division by 0.

input Domain is intersection of independent domain

Composite function

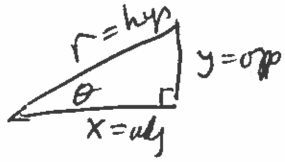
$$f\circ g(x) = f(g(x))$$

p16 eg $f(x)=\sqrt{x} \quad g(x)=x+1$

$f\circ g(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$

$g\circ f(x) = g(\sqrt{x}) = \sqrt{x}+1$

1.3 Trig.



$\sin\theta = \dfrac{y}{r} \qquad \cos\theta = \dfrac{x}{r} \qquad \tan\theta = \dfrac{y}{x}$

$\csc\theta = \dfrac{r}{y} = \dfrac{1}{\sin\theta} \qquad \sec\theta = \dfrac{r}{x} = \dfrac{1}{\cos\theta} \qquad \cot\theta = \dfrac{x}{y} = \dfrac{1}{\tan\theta}$

$\cos^2\theta + \sin^2\theta = 1 \implies 1 + \tan^2\theta = \sec^2\theta$

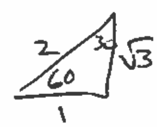
$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

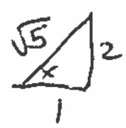
$\cos^2\theta = \dfrac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

p 28 #51 $\quad \sin^2\theta = \dfrac{3}{4} \implies \sin\theta = \dfrac{\sqrt{3}}{2} \implies$



$\theta = 60° = \dfrac{\pi}{3}$

#8 $\quad \tan x = 2$ what are $\sin x, \cos x$?



$\implies \sin x = \dfrac{2}{\sqrt{5}}, \quad \cos x = \dfrac{1}{\sqrt{5}}$

1.5 Exponential Functions

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n}$$

$$a^n \cdot a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$a^{-n} = \frac{1}{a^n}$$

$$(a^n)^m = a^{n \cdot m}$$

$$a^n b^n = (ab)^n$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

special base is $e = 2.718281828$

1.6 Logarithm

logs are inverse functions of exponential function

$$a^b = c \iff \log_a c = b$$

If base is e, we write

$$e^b = c \iff \ln c = b$$

$$\Rightarrow \quad a^b = \underline{c} = a^{\log_a \underline{c}} \quad \text{and} \quad \log_a c = \underline{\underline{b}} = \log_a a^{\underline{\underline{b}}}$$

$$\Rightarrow \quad e^b = \underline{c} = e^{\ln \underline{c}} \quad \text{and} \quad \ln c = \underline{\underline{b}} = \ln e^{\underline{\underline{b}}}$$

Properties:

$$\ln(ab) = \ln a + \ln b$$

$$\ln \frac{a}{b} = \ln a - \ln b$$

$$\ln a^b = b \cdot \ln a$$

$$\ln \frac{1}{x} = -\ln x$$

$$\ln 1 = 0$$

1.6 Inverse functions + inverse trig functions

an inverse functions "reverse" the action of the original function

$$y = 2x + 5 \qquad \text{if } x = 2 \Rightarrow y = 9$$

$$z = \frac{w-5}{2} \qquad \text{if } w = 9 \Rightarrow z = 2$$

NOTE e^x and $\ln x$ are inverse functions

a^b and $\log_a c$ are inverse functions

For trig functions

if $y = \sin x$

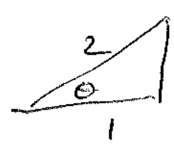
the inverse is $x = \arcsin y = \sin^{-1} y$

etc.

E.g. Given $\theta = \arccos \frac{1}{2}$

$\Longleftrightarrow \quad \frac{1}{2} = \cos\theta$

$\Rightarrow$



$\sqrt{3}$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

EX. 1.5 p 40 # 11. $16^2 \cdot 16^{-1.75}$

$$= \frac{16^2}{16^{1.75}} = 16^{.25} = 16^{1/4} = (16^{1/2})^{1/2} = 4^{1/2} = \underline{2}$$

20 $\left(\frac{\sqrt{6}}{3}\right)^2 = \frac{6}{9} = \underline{\frac{2}{3}}$

1.6 p 52 45a $e^{\ln 7.2} = 7.2$

47 a $2 \ln \sqrt{e} = \ln (\sqrt{e})^2 = \ln e = 1$

49 solve for y

$\ln y = 2t + 4 \Rightarrow e^{\ln y} = e^{(2t+4)}$

$\Rightarrow y = e^{2t+4}$

5~~6~~ a) solve for K

$e^{5k} = \frac{1}{4} \Rightarrow \ln e^{5k} = \ln \frac{1}{4}$

$\Rightarrow 5k = \ln \frac{1}{4} \Rightarrow k = \frac{1}{5} \ln \frac{1}{4}$

61 a. $5^{\log_5 7} = \underline{7}$

2.1 Rates of Change and Tangents to Curves

Slope of a line indicates how the height changes as the horiz distance changes.

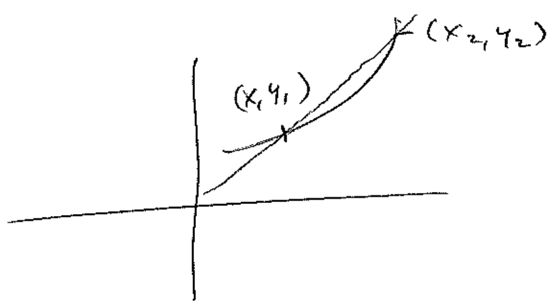
$$m = \frac{\Delta y}{\Delta x}$$

Easy to compute slope of straight line.
But change in world doesn't fit with straight line.

Given a curve $y = f(x)$, we can look at average rate of change between 2 pts on it.

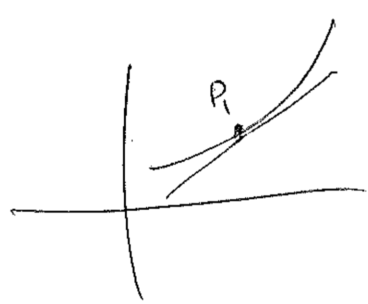
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{h}$$

This is slope of secant line through $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$.



Tangent line "touches" curve at a single pt.

slope of tangent line equals slope of secant line as $P_2 \to P_1$.



standard example of rate of change is location + speed (rate of change of location).

SCU to SLO 190 miles $= s$ (location)

if it takes 4 hours

$$\frac{\Delta s}{\Delta t} = \frac{190}{4} = 47.5 \text{ mph} \quad \text{ave}$$

if it takes 3 hours

$$\frac{\Delta s}{\Delta t} = \frac{190}{3} = 63.\overline{3} \text{ mph} \quad \text{ave.}$$

at any pt. speedometer may read more or less.

speedometer is <u>instantaneous</u> rate of change.

ave $\Leftrightarrow$ secant line

instant. $\Leftrightarrow$ tangent line

Ex 2.1 p 64

2a $g(x) = x^2 - 2x$ interval $[1, 3]$

ave rate: $\frac{\Delta g}{\Delta x} = \frac{g(3) - g(1)}{3-1} = \frac{3 - (-1)}{2} = \frac{4}{2} = 2$

7 slope at pt. $y = x^2 - 5$ $P(2, -1)$

$$\frac{\Delta y}{\Delta x} = \frac{(x+h)^2 - 5 - (x^2 - 5)}{h} = \frac{\cancel{x^2} + 2xh + h^2 - \cancel{5} - \cancel{x^2} + \cancel{5}}{h}$$

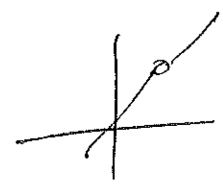
$$= \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = 2x + h \Big|_{x=2} = 4 + h$$

as h goes to 0, slope $= \underline{4}$

2.2. Limit of a Function + limit Laws.

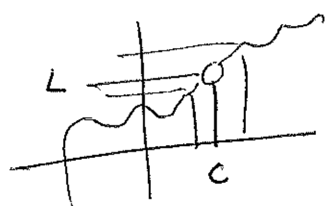
Limits are helpful when dealing with "problem" functions, but also are used with non-problem functions

Eg. $f(x) = \dfrac{x^2-1}{x-1}$



This is not defined at $x=1$.
This function has a hole in its graph.

In informal terms
$$\lim_{x \to c} f(x) = L$$
means that $f(x)$ is "close to" L when x is close to c



Limit laws are what one expects for arithmetic operation, (p 69 Th 1)
e.g. $+, -, \cdot, \div, (\;)^n, \sqrt[n]{\;}$

For polynomials (Th 2 p 70)
$$\lim_{x \to c} f(x) = f(c) \qquad \text{i.e. } \lim_{x \to 2} x^2+2x+1 = 2^2+2\cdot 2+1 = 9$$

Challenge is for quotients.
Remember $\lim_{x \to c} \dfrac{x}{x} = \lim_{x \to c} 1 = 1$

$$\therefore \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = 1+1 = 2$$

2.4 One-sided limits

It is possible to define a one-sided limit, either from the right (i.e. greater than the point) or the left (less than the point)

We write

$$\lim_{x\to c^+} f(x) \quad \text{or} \quad \lim_{x\to c^-} f(x).$$

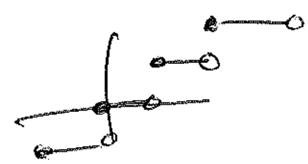
<u>Theorem</u>

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

iff

$$\lim_{x\to c} f(x) = L$$

EG $\quad y = \lfloor x \rfloor$ (floor or greatest integer function).



$$\lim_{x\to 1^+} \lfloor x \rfloor = 1$$

$$\lim_{x\to 1^-} \lfloor x \rfloor = 0 \quad \therefore \text{No limit at } x=1$$

2.5 Continuity

A function $f(x)$ is continuous at $x=c$ if

1. $f(c)$ exists (and is finite)
2. $\lim_{x\to c} f(x)$ exists (and is finite)
3. $\lim_{x\to c} f(x) = f(c)$

Eg. ① $y=x$ at $x=1$

1) $f(1)=1$
2) $\lim_{x\to 1} f(x)=1$
3) 1) = 2) $\quad\therefore$ cont's at $x=1$

② $y = \begin{cases} \dfrac{x^2-1}{x-1} & x\neq 1 \\ 3 & x=1 \end{cases}$

1) $f(1)=3$
2) $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim x+1 = 1+1 = \underline{2}$
3) 1) $\neq$ 2) $\quad\therefore$ NOT cont's at $x=1$

"cont's extension" means adjusting the value <u>at a point</u> (or <u>several</u> pts) to make a function cont's at the problem point(s).

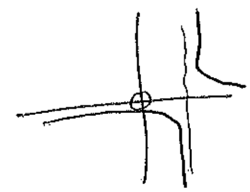
2.6 LIMITS INVOLVING INFINITY, ASYMPTOTES

BASIC $\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to \infty} \frac{1}{x} = 0$

Can use such limits to analyze certain functions, esp. "Rational functions."

If a function becomes closer & closer to a straight line as the input to the function goes to a specific x-value or to $\pm\infty$, the straight line is called an <u>asymptote</u>.

$y = \frac{1}{x-1}$ has a vertical asymptote at $x=1$ (also h. as at $y=0$)



$y = \frac{1}{x} + 1$ has a horizontal asymptote at $y=1$ (also v. asy. at $x=0$).

$$y = \frac{x^2-3}{2x-4} = \frac{x}{2} + 1 + \frac{1}{2x-4}$$

has an oblique asymptote $y = \frac{x}{2} + 1$ (and a vert asympt $x=2$)

LIMITS OF DIFF as $x \to \pm\infty$

p117 # 8 $\lim_{x \to \infty} \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right)$

$$= \lim \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right) \frac{\left(\sqrt{x^2+25} + \sqrt{x^2-1}\right)}{\sqrt{x^2+25} + \sqrt{x^2-1}}$$

$$= \lim \frac{\cancel{x^2}+25 - (\cancel{x^2}-1)}{\sqrt{x^2+25} + \sqrt{x^2-1}}$$

$$= \lim \frac{26}{\sqrt{x^2+25} + \sqrt{x^2-1}} = 0$$

3.1. Tangents & Deriv at a Pt.

Def slope of curve $y=f(x)$ at $x=x_0$ is

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \quad \text{provided limit exists}$$

Def Tangent line to y at $x=x_0$ is the ~~The~~ curve through $(x_0, f(x_0))$ with slope m.

Def The derivative of f at $x=x_0$ is the number.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

3.2 Deriv as a Function.

Def The derivative of f wrt variable x is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Ex $y = x^2 = f(x)$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim \frac{\cancel{x^2}+2x\cancel{h}+h^{\cancel{2}}-\cancel{x^2}}{\cancel{h}}$$

$$= \lim 2x+h = \underline{2x}$$

Thm If f has a deriv at $x=c$, then f is cont's at $x=c$.

7.3 Differentiation Rules

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx} c f(x) = c \frac{d}{dx} f(x)$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dy}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx} \frac{u}{v} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

p144

13 $y = (3-x^2)(x^3 - x + 1)$

$$\frac{dy}{dx} = (3-x^2)(3x^2-1) + (x^3-x+1)(-2x)$$

18 $z = \frac{4-3x}{3x^2+x}$

$$\frac{dz}{dx} = \frac{(3x^2+x)(-3) - (4-3x)(6x+1)}{(3x^2+x)^2}$$

39 $r = \frac{e^s}{s}$

$$\frac{dr}{ds} = \frac{s e^s - e^s \cdot 1}{s^2}$$

3.4 Deriv as Rate of Change.

Deriv at a point represent instantaneous rate of change.

Major example relates to change in location

$s(t)$ = location

$v = \frac{ds}{dt}$ = velocity = inst. change in locat

$a = \frac{dv}{dt}$ = acceleration = inst. change in velocity. also $\frac{d^2s}{dt^2}$

$J = \frac{da}{dt}$ = jerk = inst. change in accelert. Also $\frac{d^3s}{dt^3}$

153 #7 $s = t^3 - 6t^2 + 9t$ meters — motion of a body along s axis

$v = 3t^2 - 12t + 9$

$a = 6t - 12$

a) find accel whenever $v = 0$

$v = 0 = 3(t^2 - 4t + 3) = 3(t-3)(t-1)$

$\Rightarrow t = 1, 3$

$a|_{t=1} = 6 \cdot 1 - 12 = -6$

$a|_{t=3} = 6 \cdot 3 - 12 = 6$

b) find speed whenever $a = 0$

$a = 0 = 6t - 12 \Rightarrow t = 2$

$v = 3(4 - 8 + 3) = 3(-1) = -3 \Rightarrow$ speed $= |-3| = 3$

c) total distance traveled from $t = 0$ to $t = 2$

$s_0 = 0$

$s_1 = 1 - 6 + 9 = 4$) 4

$s_2 = 8 - 6 \cdot 4 + 18 = 26 - 24 = 2$) 2 $\Rightarrow \underline{6}$

changes direction because $v = 0$ at $t = 1$

3.5 . Derivs of Trig Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

p 160

3 $\quad y = x^2 \cos x \Rightarrow \frac{dy}{dx} = x^2(-\sin x) + (\cos x)2x$

11. $\quad y = x^2 \cos x - 2x \sin x - 2\cos x$

$$\frac{dy}{dx} = -x^2 \sin x + \cancel{2x\cos x} - \cancel{2x\cos x} - \cancel{2\sin x} + \cancel{2\sin x}$$

$$= -x^2 \sin x$$

3.6. The Chain Rule.

If $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

eg. $\quad y = \cos^2(3x^2) = [\cos(3x^2)]^2$

$$\frac{dy}{dx} = 2\cos(3x^2) \cdot (-\sin(3x^2)) \cdot 6x$$

$$= -12x \sin(3x^2) \cos 3x^2$$

168 # 35 $\quad y = xe^{-x} + e^{x^3}$

$$\frac{dy}{dx} = xe^{-x}(-1) + e^{-x} \cdot 1 + e^{x^3} \cdot 3x^2$$

3.7 Implicit differentiation

Ex 3 $\quad y^2 = x^2 + \sin xy$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(x\frac{dy}{dx} + y\right) = 2x + x\cos xy \frac{dy}{dx} + y\cos xy$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy \Rightarrow \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

3.8 Deriv of Inverse Functions + Logs

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$$

Logarithmic diff.

$$y = x^x$$

$$\Rightarrow \ln y = \ln(x^x) = x \ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cancel{x \frac{1}{x}} + \ln x = 1 + \ln x$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x)$$

3.9 Inverse Trig Functions

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$$

$$\frac{d}{dx}(\arctan e^{x^2}) = \frac{1}{1+e^{2x^2}} \cdot e^{x^2} \cdot 2x$$

3.10 Related Rates

These problems make use of the chain rule to find one rate-of-change, given another rate-of-change.

I.e. given $\frac{dx}{dt}$ and given $y=f(x)$, we want $\frac{dy}{dt}$.

Using the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

Pg 199 # 12

Cube's surface area increases at rate of $72 \text{ in}^2/\text{sec}$

At what rate is cube's volume changing when edge is 3in?

$A = 6x^2 \qquad V = x^3$

$\frac{dA}{dt} = 12x \frac{dx}{dt} \Rightarrow 72 = 12x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{6}{x}$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = 3x^2 \cdot \frac{6}{x} = 18x \Big|_{x=3} = 18 \cdot 3 = \underline{54}$

3.11 Linearization & Differentials.

The linearization of a function $y=f(x)$ at a pt. $x=a$ is the tangent line at that pt.

$$L(x)=f(a)+f'(a)(x-a)$$

At values of x near a, $f(x)\approx L(x)$.

Ex 2. $f(x)=\sqrt{1+x}$ at $x=3$

$f(x)=(1+x)^{1/2} \Rightarrow f'(x)=\frac{1}{2}(1+x)^{-1/2}$

$f(3)=4^{1/2}=2 \qquad f'(3)=\frac{1}{2}\,\frac{1}{4^{1/2}}=\frac{1}{4}$

$\therefore L(x)=2+\frac{1}{4}(x-3)=2+\frac{x}{4}-\frac{3}{4}=\frac{5}{4}+\frac{x}{4}$

Differentials are the "dy" or "dx" or "dt" term without any denominator.

EG. Given $\frac{dy}{dx}=f'(x)$

we write $dy=f'(x)\,dx$

as differentials.