SLOPES OF LINES (App A.3)

p AP-9 → AP-13

Given 2 pts on a line, (x_1, y_1), (x_2, y_2)

the slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x} = \dfrac{\text{rise}}{\text{run}}$

Given a pt (x_1, y_1) and the slope, one eq of a line is

$$\frac{y - y_1}{x - x_1} = m$$

and if the pt is $(0, b)$ (i.e., b is the y-intercept)

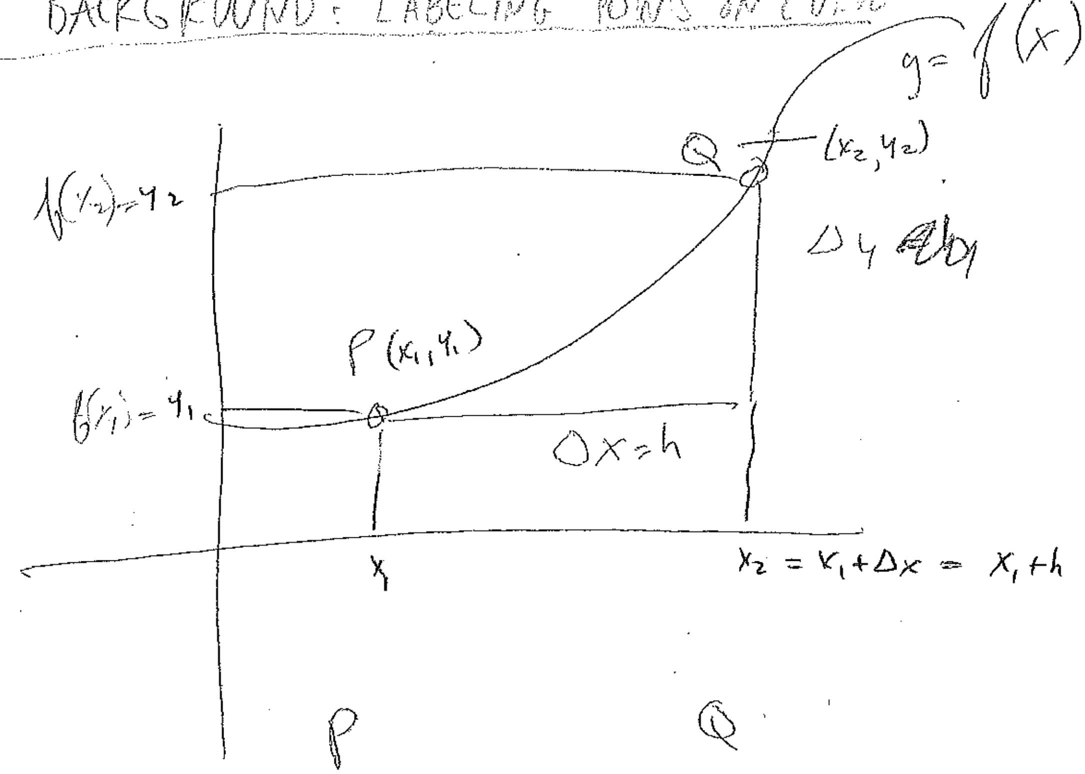
we get $\dfrac{y - b}{x} = m$

$$\Rightarrow \quad y = mx + b$$

Slope indicate how much y (height) changes as x (horizontal distance) changes.

BACKGROUND: LABELING POINTS ON CURVE

(2.1)



P	Q
(x_1, y_1) $(x_1, f(x_1))$	(x_2, y_2) $(x_2, f(x_2))$
(x_P, y_P)	(x_Q, y_Q)
(x, y)	$(x+\Delta x, y+\Delta y)$
$(x, f(x))$	$(x+\Delta x, f(x+\Delta x)) = (x+h, f(x+h))$

$$y = f(x)$$
$$y_1 = f(x_1)$$
$$y_Q = f(x_Q)$$
$$y + \Delta y = f(x + \Delta x)$$

$$\begin{aligned}
\Delta y &= y_2 - y_1 \\
&= y_Q - y_P \\
&= y + \Delta y - y \\
&= f(x+\Delta x) - f(x) \\
&= f(x+h) - f(x)
\end{aligned}$$

SLOPE OF SIMPLE CURVES (2.1) p 56 [14]

p 58 Def The secant line is the line joining 2 pts on a curve

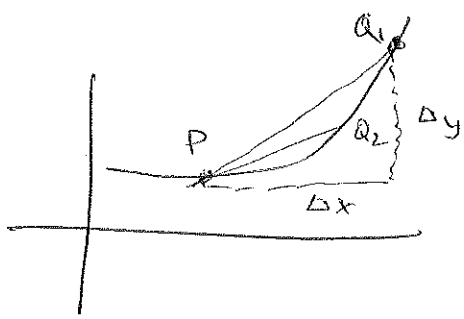
Informal Def. The tangent line is the line "touching" a curve at some given pt.

Question: How do we get the value of the slope of a curve?

Remarks: 1) It should somewhat coincide with the definition of the slope of a straight line $= m = \frac{\Delta y}{\Delta x}$

2) Since a curve changes slope depending on where you are on the curve, we must evaluate the slope at a given point.

→ The secant line serves as a general approximation to the slope of a curve.



Suppose we want the slope of the curve at point P. The slope of the secant line $\overline{PQ}$ is a general approximation $= \frac{\Delta y}{\Delta x} = m$.

As Q moves closer to P, the approximation improves

The slope of the curve at P is defined to be the slope of the tangent line at P.
The slopes of secant lines (as Q gets closer to P) become closer to the slope of the tangent line at P.

"tangent to curve calculator"

symbolab.com

$4x^2 - 4x + 1$, at $x = 1$ $y = 4x - 3$

VIA computer

google – Secant line calculator

desmos.com/calculator

why is this important?

MEASURING CHANGE

Change happens! Plants grow. Populations increase. Stock values decrease or increase. Cars change position. We'd like to be able to measure change somehow.

Change usually involves measurement over some time period. Eg., change in height or weight over a 2 year period.

Sometimes, the easiest phenomenon to use as an example of change is movement/motion.

2 prelim concepts:

AVERAGE CHANGE (over a longer period of time)

INSTANTANEOUS CHANGE (at a specific moment)

Finding the rate of

pg 58 AVERAGE change corresponds to finding the slope of a SECANT line.

Finding the rate of

p 60 INSTANTANEOUS change corresponds to finding the slope of a TANGENT line.

Defs

(14) p56 DISPLACEMENT = change in position = distance covered

$$= \Delta s = f(t+\Delta t) - f(t)$$

p56 AVERAGE VELOCITY (AVE. SPEED)

cf (2.4) 145

$$= \frac{\text{displacement}}{\text{time for travel}} = \frac{\Delta s}{\Delta t}$$

EG. $\frac{400 \text{ miles}}{8 \text{ hours}} = 50$ mph (on average)

(LA – SJ)

(slope of <u>secant</u> line)

NOTE: $\frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$

p60 INSTANTANEOUS VELOCITY

$\frac{\Delta s}{\Delta t}$ as Δt decreases to 0

EG. North of SLO – 75 mph downhill.

(slope of <u>tangent line</u> at a pt.)

EXAMPLES

NOTE: Free fall formula for feet is

$$y = 16t^2 \qquad (t = \text{time})$$

(14)

p. 56 for meters it is

$$y = 4.9t^2$$

p56 EX1 a rock breaks loose from the top of a tall cliff. what is the average speed?

b) between sec 1 and sec 2?

$\Delta t = 2-1 = 1$ sec

$$\frac{\Delta s}{\Delta t} = \frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{1}$$

$$= \frac{16 \cdot 4 - 16 \cdot 1}{1} = 16\frac{(4-1)}{1} = 16 \cdot 3 = \underline{\underline{48}} \text{ sec.}$$

p57 EX2 Find speed of falling rock in EX1 at

b) $t = 2$ sec.

NOTE: This is instantaneous speed.

I.e. this is speed when $\Delta t = 0$.

BUT we can't divide by $\underline{0}$!!

Ave. speed over interval t_0 to t_0+h can be written as

$$\frac{\Delta s}{\Delta t} = \frac{\Delta y}{\Delta t} = \frac{16(t_0+h)^2 - 16t_0^2}{h}$$

Let's do some algebra.

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0+h)^2 - 16t_0^2}{h} = \frac{16(t_0^2+2t_0h+h^2)-16t_0^2}{h}$$

$$= \frac{\cancel{16t_0^2} + 32t_0h + 16h^2 - \cancel{16t_0^2}}{h} = \frac{32t_0h+16h^2}{h}$$

$$= \frac{(32t_0+16h)h}{h} \underset{\text{need to be careful}}{=} 32t_0 + 16h$$

If $t_0 = 2$ sec and $h=0$,

$$32t_0 + 16h = 32\cdot 2 + 16\cdot 0 = \underline{64}$$

(14)

see Table 2.1 (p 57) for details when h is small.

p. 59

EX 3 Find the slope of the parabola $y=x^2$ at the point $P(2,4)$.
Write an eq. for the tangent to the parabola at that point.

Gen Secant slope $= \frac{\Delta y}{\Delta t} = \frac{(x_0+h)^2-(x_0)^2}{h} = \frac{\cancel{x_0^2}+2x_0h+h^2-\cancel{x_0^2}}{h}$

$$= \frac{2x_0h+h^2}{h} = \frac{(2x_0+h)h}{h} = 2x_0+h$$

Since we are given $x_0=2$, the secant slope $= 2\cdot 2+h = \underline{h+4}$.

As h decreases to 0, the secant slope approaches $\underline{4}$ (since secant line approaches tangent line as h goes to 0).

So $m=4$ (for tangent line) and point is $P(2,4)$

$m = \frac{y-y_0}{x-x_0}$ is pt-slope formula for line.

$\therefore \quad 4 = \frac{y-4}{x-2}$

$\Rightarrow \quad y-4 = 4(x-2)$

$\Rightarrow \quad y = 4x-8+4$

$= \underline{4x-4}$

USING LIMITS TO GET TANGENTS (3.1)

p121 '14

In 2.1, we looked at secant lines and tangent lines. We used algebra to get from an expression for a secant slope to an expression for the slope of a tangent.

Now we will look at a more precise approach using limits

DERIVATIVE AT A POINT

14
(3.1) p 128 **Def** The slope of the curve $y=f(x)$ at $P(x_0, f(x_0))$ is

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ (provided the limit exists)

The tangent line to $y=f(x)$ at P is the line through P with slope m

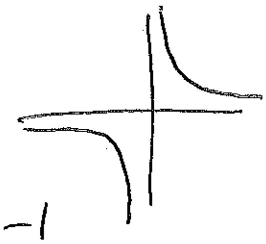
p 128 **Def** The derivative of $y=f(x)$ at x_0 (denoted $f'(x_0)$) is.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided this limit exists.

$\Longrightarrow$ The derivative enables use to find the instantaneous rate of change of a function.

EXAMPLE



EX 1 Given $y = \frac{1}{x}$

14 p 122

a) find slope at $x = a \neq 0$, slope at $x = -1$

b) where does slope equal $-\frac{1}{4}$?

c) what happens to tangent as pt $(a, \frac{1}{a})$ changes?

a) slope $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$

$= \lim \frac{\frac{a - (a+h)}{a(a+h)}}{h} = \lim \frac{-\cancel{h}}{(\cancel{h})(a)(a+h)}$

$= \lim \frac{-1}{a(a+\underset{\to 0}{h})} = \underline{\underline{\frac{-1}{a^2}}}$

$\therefore$ at $x = -1$, slope is $\frac{-1}{(-1)^2} = \frac{-1}{1} = \underline{\underline{-1}}$

b) $\frac{-1}{a^2} = -\frac{1}{4} \Rightarrow a^2 = 4 \Rightarrow a = -2, 2$

$\therefore$ pts $(-2, -\frac{1}{2}), (2, \frac{1}{2})$

c) Since a^2 is positive, $m = -\frac{1}{a^2}$ is always negative.

As $a \to \infty$ or $a \to -\infty$, $m \to 0$

As $a \to 0^+$, $m \to -\infty$

Also as $a \to 0^-$, $m \to -\infty$

EX 2
p123

Distance traveled by free-falling object
is $y = s = 16t^2$ (Sec 2.1 p56)

Instantaneous velocity at $t=1$ is the derivative of $y=s$

$$\lim_{h \to 0} \frac{16(t+h)^2 - 16t^2}{h} \Bigg|_{t=1}$$

$$= \lim \frac{16(t^2+2th+h^2) - 16t^2}{h}$$

$$= \lim \frac{\cancel{16t^2} + 32th + 16h^2 - \cancel{16t^2}}{h} = \lim \frac{+32th + 16h^2}{h}$$

$$= \lim_{h \to 0} +32t + 16h \Big|_{t=1} = 32t \Big|_{t=1} = \underline{32}$$