COS LIMIT RULE (2.4)

14

pg 87

Remember our half-angle formula

$$\sin\left(\frac{h}{2}\right) = \sqrt{\frac{1-\cos h}{2}}$$

From this we get

$$2\sin^2\frac{h}{2} = 1-\cos h$$

or $\cos h - 1 = -2\sin^2\frac{h}{2}$

$$\therefore \lim_{h\to 0}\frac{\cos h-1}{h} = \lim_{h\to 0}\frac{-2\sin^2\frac{h}{2}}{h}$$

$$= -\lim_{h\to 0}\frac{\sin^2\frac{h}{2}}{h/2}$$

$$= -\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\sin\left(\frac{h}{2}\right)$$

$$= -(1)(0) = 0$$

$$\therefore \lim_{h\to 0}\frac{\cos h-1}{h} = 0$$

INFINITY AND LIMITS

We modify slightly our definition of a limit if we let the variable tend toward either $\pm \infty$.

Def. $\lim_{x \to \pm\infty} f(x) = L$ iff.

$\forall \epsilon > 0 \quad \exists \pm M$ s.t.

whenever $M < \pm x$ then $|f(x) - L| < \epsilon$.

with $M > 0$ in either case.

Ex

$\lim_{t \to \infty} \frac{1}{t} = 0$ (i.e. as t gets bigger, $\frac{1}{t}$ becomes negligably small)

Thm 12 – Thm 1 of sec 2.2 (p 66) (sum, difference, product, quotient of limits) also holds if the variable tends toward ∞.

(p 76) 103[14]

EVALUATION METHODS FOR RATIONAL FUNCTIONS

106

① Divide quotients of polynomials by largest power.

② substitute $x = 1/h$ and take limit as $h \to 0^+$. (not in book) of EX 6 p 105

pg 104

Ex method 1

$$\lim_{x \to \infty} \frac{2x^2 - x + 3}{3x^2 + 5} = (\text{dividing by } x^2) \lim_{x \to \infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{5}{x^2}} = \frac{2}{3}$$

Ex method 2

$$\lim_{x \to \infty} \frac{2x^2 - x + 3}{3x^2 + 5} = (\text{substituting}) \lim_{h \to 0^+} \frac{2\frac{1}{h^2} - \frac{1}{h} + 3}{3\frac{1}{h^2} + 5} = (\text{algebra}) \lim_{h \to 0^+} \frac{2 - h + 3h^2}{3 + 5h^2}$$

$$= \frac{2}{3}$$

INFINITE LIMITS (p107)[14]

EG.

$$\lim_{t \to 0^+} \frac{1}{t} = \infty .$$

This makes sense.

"as t gets smaller, $\frac{1}{t}$ gets large without bounds."

However, beware of "tricks" of arithmetic.

$$\infty + \infty = \infty$$

But $\infty - \infty$ doesn't make sense, neither does $\frac{\infty}{\infty}$!

EG.

PS ~~94 # 114~~ ~~NOT IN PR ed.~~ cf EX9 pg 106

~~7.8 (90 # 69a)~~

$$\lim_{n \to \infty} \sqrt{n^2+1} - n$$

If we try to "plug in" ∞, we get $\infty - \infty$.

$\therefore$ we use this trick – "rationalize" the expression by multiplying by 1,

written as $\dfrac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n}$

We get.

$$\lim_{n \to \infty} \left(\sqrt{n^2+1} - n\right) \frac{\left(\sqrt{n^2+1} + n\right)}{\left(\sqrt{n^2+1} + n\right)} = \lim \frac{\cancel{n^2} + 1 - \cancel{n^2}}{\sqrt{n^2+1} + n}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n^2+1} + n} = 0 .$$

ASYMPTOTES

(14) Pg 104 An *asymptote* is a *straight line* which a curve gets closer to as it moves in a given direction.

A curve may have NO asymptotes or *many* asymptotes.

The asymptotes may be *horizontal* (∥ to x-axis), *vertical* (∥ to y-axis) or *oblique* (not parallel to either axis). p 107

Def Given a curve $y = f(x)$, if $\exists$ a line $y = mx + b = p(x)$ s.t. as $x \to c$ then $|f(x) - p(x)| \to 0$

Then $p(x)$ is an *asymptote* of curve $f(x)$.

(14) p 110 **Def.** A line $x = a$ is a *vertical asymptote* if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

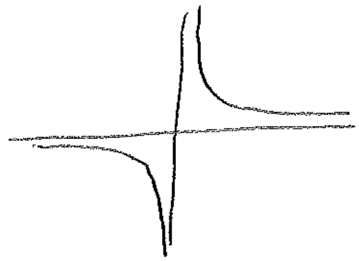
p 104 **Def** A line $y = b$ is a *horizontal asymptote* if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

EXAMPLES

(1) $y = \frac{1}{x}$ asymptotes : x-axis, y-axis.

(14) cf EX 1 p103

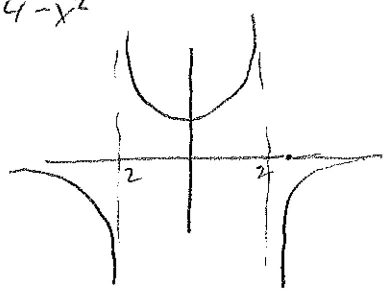


as $x \to 0$, $y \to \infty$
$\therefore$ $x = 0$, i.e. y-axis is vertical asymptote

as $x \to \infty$, $y \to 0$
$\therefore$ $y = 0$, i.e. x-axis is horiz. asymp.

(2) $y = \frac{8}{4 - x^2} = \frac{-8}{x^2 - 4}$

EX 17 p111



as $x \to \pm 2$, $y \to \pm \infty$
i.e. $x = \pm 2$ are 2 vertical asymp.

as $x \to \pm\infty$ $y \to 0$
i.e. x-axis is horiz. asymp.

(3). $y = \frac{x^2 - 3}{2x - 4}$

EX 10 p107

try dividing

$$
\begin{array}{r}
\frac{1}{2}x + 1 \\
2x - 4 \,\big)\, \overline{x^2 - 3} \\
x^2 - 2x \\
\hline
2x - 3 \\
2x - 4 \\
\hline
1
\end{array}
$$

$$y = \frac{1}{2}x + 1 + \frac{1}{2x - 4}$$

now as x gets larger, the last term gets smaller and y approximates the line defined by the 1st 2 terms

$\frac{1}{2}x + 1$.

DOMINANT TERMS

(p41)
p111

from last example we know

$$y = \frac{x^2-3}{2x-4}$$

also can be written as

$$y = \frac{1}{2}x + 1 + \frac{1}{2x-4}$$

Depending on which x-values are chosen, the y-value may be primarily due to only some of the Terms and not others.

For example.

as $x \to \infty$, $\frac{1}{2x-4}$ is negligible

$\therefore \quad y \approx \frac{1}{2}x + 1$ and

we say "$\frac{1}{2}x+1$ __dominates__" (or is the __dominant term__).

__for x large__

as $x \to 2$, $\frac{1}{2x-4}$ is very large and

$\frac{1}{2}x + 1 \to 2$ which is negligible in comparison.

$$\therefore \quad y \approx \frac{1}{2x-4}$$

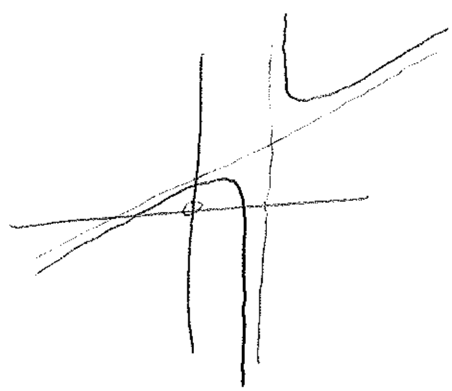
and we say "$\frac{1}{2x-4}$ __dominates__" for x close to 2.

One the other hand, as $x \to 2$,

$$y = \frac{x^2-3}{2x-4} \to \frac{4-3}{4-4} \to \infty$$

$\therefore$ oblique asymp. $y = \frac{1}{2}x + 1$

vertical asymp $x = 2$



USES

One can make use of symmetries + asymptotes when analyzing curves, to aid in graphing, for example.

Asymptotes can aid in giving the boundary limits of a curve. Symmetries can help.
If one's picture does not match, then one's analysis is probably wrong.