6–1 21

(87–7)

88–7

MORE ON LIMITS

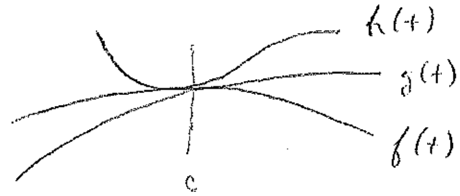
① "SANDWICH THM"

2.2 Thm 4 p69 '14 **Thm** Suppose $f(t) \le g(t) \le h(t)$ for all t around a given point c.

Suppose $f(t) \to L$ and $h(t) \to L$ as $t \to c$

then $g(t) \to L$ as $t \to c$

In other words.



h + f are the bread,

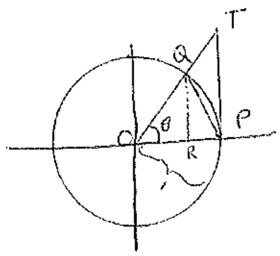
g is the meat — if g is between f and h, then if f + h reach the same limit at c g must do the same.

② TRIG. AND LIMITS.

2.4 p86 '14 ⁷[Ex 12, p 66] ⁸(Ex 3, pg 81)

Th 7 p86 **Thm.** Let θ be in radians, then $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

proof



NOTES: ① $OP = OQ = 1$

② $\sin\theta = \frac{QR}{OQ} = \frac{TP}{OT}$

or $OQ\sin\theta = QR = 1 \cdot \sin\theta$

③ 2π radians $= 360°$

area circle $= \pi r^2$

∴ area sector of unit circle $= \frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}$

④ $\tan\theta = \frac{TP}{OP} = \frac{TP}{1} = TP$

From diagram

$$\underset{Ⓐ}{\text{area }\triangle OPQ} < \underset{Ⓑ}{\text{area sector } OPQ} < \underset{Ⓒ}{\text{area }\triangle OPT}$$

Ⓐ area $\triangle OPQ = \frac{\text{base} \cdot \text{height}}{2} = \frac{1}{2}(OP)(QR) = \frac{1}{2}(1)(\cancel{OQ}\sin\theta) = \frac{\sin\theta}{2}$

Ⓑ area sector $OPQ = \frac{\theta}{2}$ by note #3

Ⓒ area $\triangle OPT = \frac{1}{2}(OP)(PT) = \frac{1}{2} \cdot 1 \cdot \tan\theta = \frac{\tan\theta}{2}$ (note 4)

$$\therefore \quad \underset{(A)}{\frac{\sin\theta}{2}} < \underset{(B)}{\frac{\theta}{2}} < \underset{(C)}{\frac{\tan\theta}{2}} \Rightarrow \sin\theta < \theta < \tan\theta$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \quad \Rightarrow \quad 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

By Sandwich Thm above, both 1 and $\cos\theta \to 1$ as $\theta \to 0$

∴ the middle goes to 1 as well i.e. $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$ //QED

EXAMPLES

① $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right)\left(\frac{1}{\cos \theta}\right) = 1 \cdot 1 = 1$

② $\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} 2 \cdot \frac{\sin(2x)}{(2x)} = 2 \cdot \lim_{x \to 0} \frac{\sin(2x)}{(2x)}$

$= 2 \cdot 1 = 2$

(2,5)(14) p 90 (92-7) 6-3

A CONTINUITY [redacted] via

(88-8)

(Test) p 85

p92

Def Let f be a function. Let c be a given point and let f be defined in some area on both sides of c (i.e. $\exists$ some a and b, s.t. f is defined on (a,c) and (c,b)).

If a) f is defined at c, i.e. $f(c)$ exists and is finite,

b) $\lim_{x \to c} f(x)$ exists and is finite,

c) $\lim_{x \to c} f(x) = f(c)$

then f is said to be continuous at c (interior point of an interval (a,b))

(abbrev. conts, cnts or similar).

- (at end points we use appropriate 1-sided limits)

Ex 1

$f(x) = x^2+1$ at $x=2$ is cnts

a) $f(2) = 5$

b) $\lim_{x\to 2} f(x) = \lim_{x \to 2} x^2+1 = 5$

c) $5=5$ $\therefore$ cnts.

Ex 2a

$$f(x) = \frac{x^2-4}{x-2} \quad \text{at } x=2$$

a) $f(2)$ is not defined at 2 $\therefore$ not continuous.

Ex 2b

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{at } x=0$$

a) $f(0)$ is not finite $\therefore$ not cont's.

Ex 3a

$$f(x) = \begin{cases} -2 & x<3 \\ 1 & x=3 \\ 10 & x>3 \end{cases} \qquad \text{at } x=3$$

a) $f(3)=1$

b) no limit exists since right + left sided limits are not equal

$\therefore$ not cont's.

Ex 3b

$$f(x) = \begin{cases} \left|\frac{1}{x}\right| & x \neq 0 \\ 0 & x=0 \end{cases} \qquad \text{at } 0$$

a) $f(0)=0$

b) $\lim_{x\to 0} f(x) = \infty$ not finite. $\therefore$ not cont's.

Ex 4

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & x \neq 3 \\ 2 & x=3 \end{cases} \qquad \text{at } 3$$

a) $f(3)=2$

b) $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = 6$ (14)

p 99

c) $2 \neq 6$ $\therefore$ not equal $\therefore$ not cont's.

NOTE: If we re-defined f s.t. $f(3)=6$, it would be cont's. i.e. the discontinuity was "removed", (p 85) and the new f would be the "continuous extension" of the old f. (p. 98)

p 93

COMMENT: Basic intuitive concept of continuity: if $f(x)$ has a hole or a jump in it at a given point, it is not cont's at that point.

[NOT always possible!]

B DISCONTINUITY

Sometimes it helps to look at the opposite in order to understand a concept. (14)

EX 3/ pg 92

① Greatest Integer function $f(x) = [x]$ – discontinuous at all integers.

② $$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \quad \text{for } x \in [0,1]$$

discontinuous at all points of the interval.

C SETS and COMBINED FUNCTIONS

<u>Def</u> If f is cont's on every point in a set A, then f is said to be <u>cont's on A</u>.

<u>Def</u> If f is cont's on some set, but the set is unknown, obvious, or unimportant, then f is said to be <u>cont's</u> (or <u>cont's somewhere</u>).

<u>Def</u> If f is cont's on the entire real line, then f is said to be <u>cont's everywhere</u>.

8, p 94 <u>Thm</u> Let f, g be cont's at $x = c$ and K be any number. then

i) $f(x) + g(x)$

ii) $f(x) - g(x)$

iii) $f(x) \cdot g(x)$

iv) $K \cdot g(x)$

v) $\dfrac{f(x)}{g(x)}$ if $g(c) \neq 0$ + powers + roots

are all cnts at $x = c$.

p 95 '14

Thm 9 If f is cont's at c, and g is cont's at $f(c)$,

then the composite function $g \circ f = g(f(x))$ is also cont's at $x = c$.

EX

$x^2 + 3x^5$ is conts (i)

$3x^5$ is conts (iv)

$(x+1)^2$ is conts (comp.) — $f(x)$ is $x+1$, $g(w) = w^2$

$(x+1)^2(3x-5)^3$ is conts (iii)

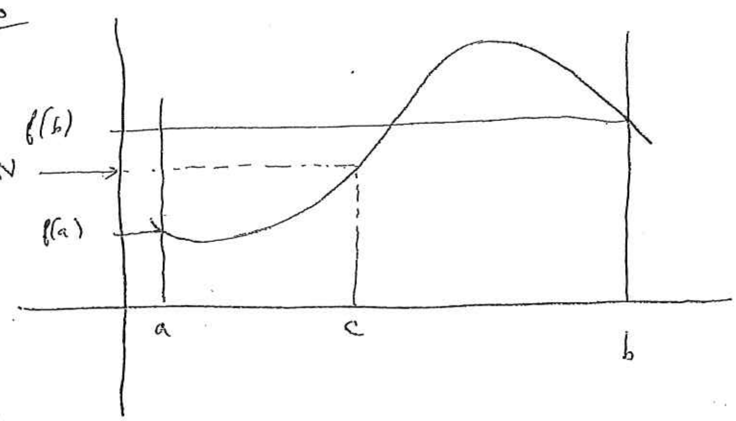
x_1 x_2

p 97 '14

Thm 11 (Intermediate Value Thm) Let f be cont's on a closed interval $[a, b]$. Let N be any number s.t. $f(a) \le N \le f(b)$.

Then $\exists\, c$ s.t. $a \le c \le b$ and $f(c) = N$. I.e. $f(x)$ takes on every possible value between $f(a)$ and $f(b)$.

EG



1(2–7)

EX

~~1970 H 39~~

~~87~~

7^

14 pg 107 #55

f is conts. at $x=0$ $f(0)<0$

at $x=1$ $f(1)>0$

Why does it have one solution (at least) between $x=0$ and $x=1$?

If $x=c$ is a solution, then $f(c)=0$

and 0 is between a negative and positive number.

By Interm. Val. Thm (Thm) $\exists\, c$ between 0 and 1

s.t. $f(c)=0$.