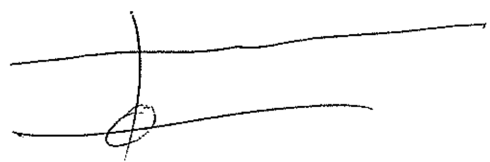
PRELIM

Plot

$$y = 1$$

(1pt. is it a function?)



Plot $y = \frac{x-1}{x-1}$

INTUITIVE DEF OF LIMIT

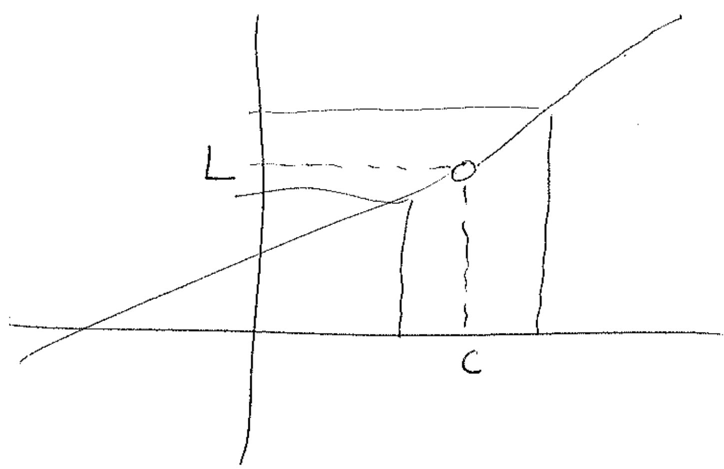
We want to say:

① $f(x)$ is defined around the point in question –
i.e. if point is $x = c$, f is defined on (t_1, c) and (c, t_2)

② if you take a value of x "close" to c,
Then $f(x)$ is "close to" the limit value L.

In This case we write

$$\lim_{x \to c} f(x) = L$$



Limits are a way of saying a function "should have" the value of L at $x = c$, even though, <u>for some functions</u>, the function may not even be defined there.

Start with L-value. If, no matter how close we are to L on y-axis, we can find an interval close to c on x-axis, Then L is The limit of $f(x)$ as $x \to c$.

E INTRODUCTION TO LIMITS

On occasion, functions may not be defined at a given point, even though they are defined in the area surrounding that point, such as $f(x) = \frac{x^2-1}{x-1}$.

This is NOT defined at $x=1$, since then $f(x)$ would have to be $\frac{0}{0}$ which doesn't make mathematical sense.

However, everywhere else, $f(x)$ is defined, no matter how close you get to $x=1$. Algebraic simplification also shows us that $f(x)$ is the same as $g(x)=x+1$, at all x s.t. $x \neq 1$.

We would like to have some way to say that $f(1)$ "should be" 2, even though it really isn't.

This leads to the concept of limit.

For example, we want to say,

"$f(x)$ has a limit value of 2, as x approaches 1."

We write this.

$$\lim_{x \to 1} f(x) = 2$$

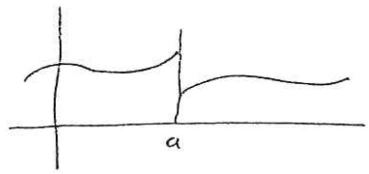
or

$$f(x) \to 2 \quad \text{as} \quad x \to 1$$

E Problem Curves

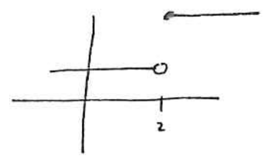
Note: Not every line or curve has a limit value at every point.
There are no problems with normal "continuous" curves.
We start getting problems with curves that "jump."

EG (#1)



This has no limit at a, since one can pick an error tolerance (ϵ) around L s.t. $\nexists\ \delta$ to satisfy the definition.

(#2) Look at line $y = \begin{cases} 1 & x < 2 \\ 3 & x \geq 2 \end{cases}$



No limit

Say $\lim_{x \to 2} y = 3 = L$

Let $\epsilon = \frac{1}{2}$. Pick any δ.

Look at the point $2 - \frac{\delta}{2} = t$

Now $|t - c| = \left|2 - \frac{\delta}{2} - 2\right| = \left|-\frac{\delta}{2}\right|$ which is $< \delta$

However $f(t) = f\left(2 - \frac{\delta}{2}\right) = 1$

$\therefore |f(t) - L| = |1 - 3| = |-2| = 2 \not< \epsilon = \frac{1}{2}$

$\therefore$ 3 is not the limit and we can easily show 1 is not limit either.

C EVALUATING SIMPLE LIMITS

First look at 2 functions that are ~~not~~ problems to evaluate them. (3a) p 67

(1) $f(x) = x$ (cf EX 3a p 67) $\Rightarrow \lim_{x \to c} f(x) = c$

Let $\epsilon > 0$. We want to show $\exists \delta$ s.t. if $|t - c| < \delta$

then $|f(t) - c| < \epsilon$

But $f(t) = t$ $\therefore$ let $\delta = \epsilon$

$\therefore$ if $|t-c| < \delta$ then $|t-c| < \epsilon = \delta$ or $|f(t) - c| < \epsilon$ // QED.

$\Rightarrow \lim_{x \to 2} x = 2 \qquad \lim_{z \to 4} z = 4$

(2) $f(x) = \begin{cases} K & x \neq c \\ ? & x = c \end{cases} \Rightarrow \lim_{x \to c} f(x) = K$

(cf EX 3b p 67)

Let $\epsilon > 0$. We want to show $\exists \delta$ s.t. if $|t - c| < \delta$

then $|f(t) - 1| < \epsilon$

But $\forall t \neq c$, we have $f(t) = 1$

$\therefore |f(t) - 1| = |1 - 1| = 0 < \epsilon \quad \forall t \neq c$

$\therefore$ no matter what δ is,

if $|t - c| < \delta$, then $|f(t) - 1| < \epsilon$

$\therefore \lim_{x \to c} f(x) = 1$ // QED

$\Rightarrow \lim_{x \to 5} 1 = 1 \quad , \quad \lim_{x \to 10} \pi = \pi$

D USES

E

LIMITS are specially useful for functions that are "almost" continuous except for a point (or a few points)

E.G. $f(x) = \dfrac{x^2-1}{x-1}$

Frequently these "almost" continuous functions are quotient functions where the denominator will go to 0 if evaluated at some point.

LIMITs can be useful in dealing with such functions because of the following Thm.

P5 ~~14B~~ ~~7M~~ ~~69~~ 66

Thm 1 Let $\lim_{t\to c} f_1(t) = L_1$ and $\lim_{t\to c} f_2(t) = L_2$

If both limits exist and are finite, then

i + ii) $\lim (f_1(t) \pm f_2(t)) = \lim f_1(t) \pm \lim f_2(t) = L_1 \pm L_2$

iii) $\lim (f_1(t) \cdot f_2(t)) = (\lim f_1(t)) \cdot (\lim f_2(t)) = L_1 \cdot L_2$

iv) $\lim K \cdot f_1(t) = K (\lim f_1(t)) = K \cdot L_1$

v) $\lim \dfrac{f_1(t)}{f_2(t)} = \dfrac{\lim f_1(t)}{\lim f_2(t)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$

+ power + root rules

Using the 2 simple functions above and i – iv, we get

P5 ~~W~~ 7M P 67

Thm 2 If $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$ is any polynomial

then $\lim_{t\to c} f(t) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0$

E EXAMPLES

① $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x \to 1} \left(\dfrac{x-1}{x-1} \cdot x+1 \right) \overset{iii}{=} \lim\limits_{x \to 1} \left(\dfrac{x-1}{x-1} \right) \lim\limits_{x \to 1} (x+1)$

$= 1 \cdot \lim\limits_{x \to 1} (x+1) = 1 \cdot 2 = 2$

② $\lim\limits_{x \to 2} 5x^2+3 \overset{\text{Thm } 2}{=} 5(2)^2+3 = 5 \cdot 4 + 3 = 23$

~~HW~~

~~B 71-73~~

~~17, 18, 31, 32, 39~~

~~B 73 # 21~~

(87 ... 5

88 5

A LIMIT DEF

Def (again) $\lim_{t \to c} f(t) = L$ iff

$\forall \epsilon > 0 \; \exists \delta_\epsilon > 0$ s.t.

when $|t-c| < \delta_\epsilon$ then $|f(t) - L| < \epsilon$

B. ONE-SIDED LIMITS

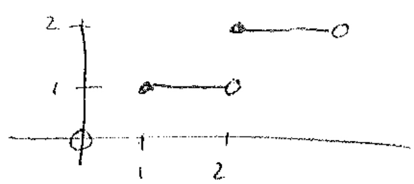
(2.4) p 83

Sometimes it is helpful to talk about "one-sided" limits. This is an intuitive concept which works + you can trust.

Look at the greatest integer function.

— at any integer, it has no limit, although it does have a value. But it does have one-sided limits, i.e. you can say what the value "should be" coming from either side going to a given t-value.

E.G. graph of $[x]$ around $x=2$



— As you approach 2 from the left or negative side, the value "should be" 1 at $x=2$

we write this $\lim_{x \to 2^-} [x] = 1$

— However, as you approach 2 from the right or positive side the value "should be" 2 at $x=2$

we write this $\lim_{x \to 2^+} [x] = 2$

C DEFINITIONS

Def 1 "Left-sided limit"

$\lim_{t \to c^-} f(t) = L$ iff

$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0$ s.t.

if $0 < c - t < \delta_\epsilon$ then $|f(t) - L| < \epsilon$.

Def 2 "Right-sided limit"

$\lim_{t \to c^+} f(t) = L$ iff

$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0$ s.t.

if $0 < t - c < \delta_\epsilon$ then $|f(t) - L| < \epsilon$.

Sometimes a limit is defined in terms of the one-sided limits and is then called a "two-sided" limit

Thm [illegible] 6, p 84 "

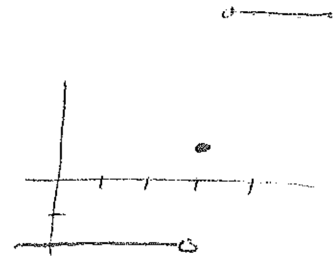
$\lim_{t \to c} f(t) = L$ iff $\lim_{t \to c^-} f(t) = \lim_{t \to c^+} f(t) = L$.

— This can provide an easy way to check if a limit exists at a point by checking whether the 1-sided limits are equal at that point.

— For example, we can conclude that the greatest integer function has no (2-sided) limit at any integer since the right and left sided limits are not the same.

D EXAMPLE

Let $f(x) = \begin{cases} -2 & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ 10 & \text{if } x > 3 \end{cases}$



Now $\lim_{x \to 3^-} f(x) = -2$

and $\lim_{x \to 3^+} f(x) = 10$

$f(3) = 1$

This function has NO (2-sided) limit since the 2 one-sided limits are not equal.

(another example – see book

14th pg 84 Ex 2

(skip ε-δ proof ... A=3 ... A=10)