ROOT FINDING: NEWTON

(4, 7) p276

17-1
16-4
88-17
16C-1

Given an equation in one variable, if we write it in the form $f(x)=0$, then the values of x which make this true are called the roots (or zeroes) of the equation.

E.G. Given $x^2-2x-3=0$, the roots are
$x = 3, -1$

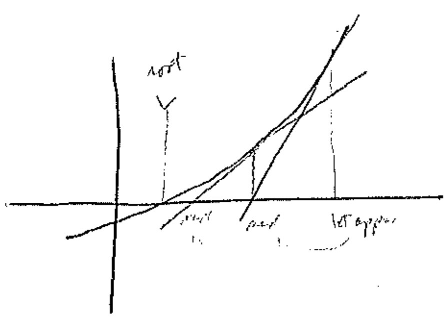
For linear equations, there is no problem to find the root.
eg $ax+b=0 \Rightarrow ax=-b \Rightarrow x=-\frac{b}{a}$

For quadratics, we use the quadratic formula
$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

There are NO easy formulas for higher order (none exist for 5^{th} or higher)

∴ We frequently use methods which give us an approximation to the answer & then give us a better & better approximation. This procedure is called iteration or an iterative method.

One common method, called Newton's Method, is based on derivatives.



We use the tangent line to find an approximate root & then try again.

~~17-2~~
16-5
16C-2

pick first guess x_1

tangent line to $f(x)$ at x_1 is.

$$\frac{y-y_1}{x-x_1} = f'(x_1) = \frac{y-f(x_1)}{x-x_1}.$$

We want to find x^* s.t. $y=0$ i.e. where this line cross x-axis.

we get $f'(x_1) = -\dfrac{f(x_1)}{x^*-x_1}$

$$\Rightarrow \quad x^*-x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow \quad x^* = x_1 - \frac{f(x_1)}{f'(x_1)}$$

New x^* is based on old x_1

we repeat this formula until x doesn't change.

This is very good for computer applications.

One can find the square root to over 100 digits in about 7 iterations!

Problem: it doesn't work for certain odd ball functions,

eg. Fig 4,53 p279

In This case $x_3 = x_1$ and $x_4 = x_2$ and approximations just repeat.

EG Find $\sqrt{2}$.

This is the same as finding root of $x^2 - 2 = 0$

$$f(x) = x^2 - 2$$
$$f'(x) = 2x$$

Formula now is $\quad x_{new} = x_{old} - \dfrac{f(x_{old})}{f'(x_{old})}$

$$x^* = x - \frac{x^2 - 2}{2x}$$
$$= \frac{2x^2 - x^2 + 2}{2x} = \frac{x^2 + 2}{2x}$$

For 1st guess, let $x = 1$

$$x^* = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Try again

$$x^* = \frac{1.5^2 + 2}{3} = \frac{2.25 + 2}{3} = \frac{4.25}{3} = 1.41666\ldots$$

Try again

$$x^* = \frac{1.41666^2 + 2}{2 \cdot 1.41666\ldots} = \frac{2.006944 + 2}{2.83333} = \frac{4.006944\ldots}{2.83333\ldots}$$

$$= 1.414215686$$

By calculator $\sqrt{2} = 1.414213562$

same to 6 digits in 3 tries!